ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0$ and $x+y+z=3$ then prove that :

$$\frac{1}{1+xy}+\frac{1}{1+yz}+\frac{1}{1+zx}\ge\frac{6}{\sqrt{x}+\sqrt{y}+\sqrt{z}}-\frac{1}{2}$$

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$$\sum_{cyc}\frac{1}{1+yz}=\sum_{cyc}\frac{1+yz-yz}{1+yz}=3-\sum_{cyc}\frac{yz}{1+yz}\overset{AM-GM}{\ge}3-\sum_{cyc}\frac{yz}{2\sqrt{yz}}$$

$$=3-\frac{1}{2}\sum_{cyc}\sqrt{yz}\overset{?}{\ge}\frac{6}{\sum_{cyc}\sqrt{x}}-\frac{1}{2}\Leftrightarrow\frac{7-\sum_{cyc}bc}{2}\overset{?}{\ge}\frac{6}{\sum_{cyc}a}\ (a=\sqrt{x},b=\sqrt{y},c=\sqrt{z})$$

$$\Leftrightarrow\frac{\frac{7}{3}\left(\sum_{cyc}a^2\right)-\sum_{cyc}ab}{2}\overset{?}{\ge}\frac{2\left(\sum_{cyc}a^2\right)}{\sum_{cyc}a}\cdot\sqrt{\frac{\sum_{cyc}a^2}{3}}\left(\because\sum_{cyc}x=3\Leftrightarrow\sum_{cyc}a^2=3\right)$$

$$\Leftrightarrow\left(7\sum_{cyc}a^2-3\sum_{cyc}ab\right)^2\left(\sum_{cyc}a\right)^2\underset{(\bullet)}{\overset{?}{\ge}}48\left(\sum_{cyc}a^2\right)^3\left(\begin{array}{c}\because\frac{7}{3}\left(\sum_{cyc}a^2\right)\ge\\ \frac{7}{3}\left(\sum_{cyc}ab\right)>\sum_{cyc}ab\end{array}\right)$$

Assigning $b+c=X, c+a=Y, a+b=Z\Rightarrow X+Y-Z=2c>0, Y+Z-X=2a>0$ and $Z+X-Y=2b>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); and then : $a=s-X, b=s-Y, c=s-Z\therefore\sum_{cyc}a=s,\sum_{cyc}ab=4Rr+r^2$,

$\sum_{cyc}a^2=s^2-8Rr-2r^2$ and hence, $(\bullet)\Leftrightarrow s^6+(200Rr+50r^2)s^4-$

$$r^2(4592R^2+2296Rr+287r^2)s^2+384r^3(4R+r)^3\underset{(*)}{\overset{?}{\ge}}0 \text{ and }\because$$

$(s^2-16Rr+5r^2)^3\overset{Gerretsen}{\ge}0\therefore$ in order to prove $(*)$, it suffices to prove :

LHS of $(*)\overset{?}{\ge}(s^2-16Rr+5r^2)^3\Leftrightarrow(248R+35r)s^4-$
$r(5360R^2+1816Rr+362r^2)s^2+r^2(28672R^3+14592R^2r+5808Rr^2+259r^3)$
$\underset{(**)}{\overset{?}{\ge}}0$ and $\because(248R+35r)(s^2-16Rr+5r^2)^2\overset{Gerretsen}{\ge}0\therefore$ in order to prove $(**)$,

it suffices to prove : LHS of $(**)\overset{?}{\ge}(248R+35r)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow(322R^2-397Rr-89r^2)s^2\underset{(***)}{\overset{?}{\ge}}r(4352R^3-5664R^2r-651Rr^2+77r^3)$$

$$\&\ (322R^2-397Rr-89r^2)s^2\overset{Gerretsen}{\ge}(322R^2-397Rr-89r^2)(16Rr-5r^2)$$

$$\overset{?}{\geq} \mathbf{r}(4352\mathbf{R}^3 - 5664\mathbf{R}^2\mathbf{r} - 651\mathbf{R}\mathbf{r}^2 + 77\mathbf{r}^3) \Leftrightarrow 400\mathbf{t}^3 - 1149\mathbf{t}^2 + 606\mathbf{t} + 184 \overset{?}{\geq} 0$$

$$\left(\mathbf{t} = \frac{\mathbf{R}}{\mathbf{r}}\right) \Leftrightarrow (\mathbf{t}-2)\big((\mathbf{t}-2)(400\mathbf{t}+451)+810\big) \overset{?}{\geq} 0 \to \textbf{true} \because \mathbf{t} \overset{\textbf{Euler}}{\geq} 2$$

$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \textbf{ is true } \forall \Delta \, \mathbf{XYZ}_{\mathbf{s},\mathbf{R},\mathbf{r}} \Rightarrow (\bullet) \textbf{ is true}$$

$$\therefore \frac{1}{1+xy} + \frac{1}{1+yz} + \frac{1}{1+zx} \geq \frac{6}{\sqrt{x}+\sqrt{y}+\sqrt{z}} - \frac{1}{2}$$

$$\forall x, y, z > 0 \,\big|\, x + y + z = 3, '' ='' \textbf{ iff } x = y = z = 1 \text{ (QED)}$$