ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \geq 0$ and $ab + bc + ca > 0$ then prove that :

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2} \geq \frac{10}{(a+b+c)^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

When exactly 1 variable equals to zero and WLOG we may assume $a = 0$ $(b, c > 0)$, then, inequality becomes : $\frac{1}{b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2} \overset{?}{\geq} \frac{10}{(b+c)^2}$

$$\Leftrightarrow b^6 + 2b^5c - 6b^4c^2 + 6b^3c^3 - 6b^2c^4 + 2bc^5 + c^6 \overset{?}{\geq} 0$$

$\Leftrightarrow (b-c)^2(b^4 + 4b^3c + b^2c^2 + 4bc^3 + c^4) \overset{?}{\geq} 0 \rightarrow$ true and we now focus on the case when : $a, b, c > 0$ and assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z$, $y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say); then : $\sum_{cyc} a = s, abc = r^2s$,

$$\sum_{cyc} ab = 4Rr + r^2, \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2, \sum_{cyc} a^2b^2 = r^2((4R+r)^2 - 2s^2)$$

and then : $\sum_{cyc} \frac{1}{b^2+c^2} \overset{?}{>} \frac{10}{(a+b+c)^2}$

$$\Leftrightarrow \left(\left(\sum_{cyc} a^2\right)^2 + \sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a\right)^2 \overset{?}{>} 10\left(\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) - a^2b^2c^2\right)$$

$$\Leftrightarrow s^2\left((s^2 - 8Rr - 2r^2)^2 + r^2((4R+r)^2 - 2s^2)\right) \overset{?}{>} 10\left((s^2 - 8Rr - 2r^2)(r^2)((4R+r)^2 - 2s^2) - r^4s^2\right)$$

$$\Leftrightarrow s^6 - (16Rr - 14r^2)s^4 - r^2(80R^2 + 200Rr + 35r^2)s^2 + 20r^3(4R+r)^3 \underset{(*)}{\overset{?}{>}} 0$$

and $\because (s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*) \overset{?}{>} (s^2 - 16Rr + 5r^2)^3 \Leftrightarrow (32R - r)s^4 - r(848R^2 - 280Rr + 110r^2)s^2 + r^2(5376R^3 - 2880R^2r + 1440Rr^2 - 105r^3) \underset{(**)}{\overset{?}{>}} 0$

and $\because (32R - r)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**) \overset{?}{>} (32R - r)(s^2 - 16Rr + 5r^2)^2$

$$\Leftrightarrow (44R^2 - 18Rr - 25r^2)s^2 \underbrace{\overset{?}{>}}_{(***)} r(704R^3 - 624R^2r - 120Rr^2 + 20r^3) \text{ and}$$

$$\text{finally, LHS of } (***) \overset{\text{Gerretsen}}{\geq} (44R^2 - 18Rr - 25r^2)(16Rr - 5r^2) \overset{?}{>} \text{RHS of } (***)$$

$$\Leftrightarrow r^2(116R^2 - 190Rr + 105r^2) \overset{?}{>} 0 \to \text{true (strict inequality)} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true } \forall \Delta XYZ_{s,R,r} \therefore \sum_{\text{cyc}} \frac{1}{b^2 + c^2} > \frac{10}{(a + b + c)^2}$$

whenever $a, b, c > 0$ and combining all cases, we have :

$$\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} \geq \frac{10}{(a + b + c)^2} \; \forall a, b, c \geq 0 \,\middle|\, ab + bc + ca > 0,$$

$$'' ='' \text{ iff } \begin{pmatrix} a = 0 \\ b = c > 0 \end{pmatrix} \text{ or } \begin{pmatrix} b = 0 \\ c = a > 0 \end{pmatrix} \text{ or } \begin{pmatrix} c = 0 \\ a = b > 0 \end{pmatrix} \text{ (QED)}$$