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If $a, \mathrm{b}, \mathrm{c} \geq 0$ and $a+\mathrm{b}+\mathrm{c}=2$ then prove that :

$$a^2+\mathrm{b}^2+\mathrm{c}^2 \geq 2(a^3\mathrm{b}^3+\mathrm{b}^3\mathrm{c}^3+\mathrm{c}^3a^3+4a^2\mathrm{b}^2\mathrm{c}^2)$$

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When exactly 2 variables equal to zero and WLOG we may assume $\mathrm{b}=\mathrm{c}=0$ $(a=2)$, then : LHS $=4>0=$ RHS

When exactly 1 variable equals to zero and WLOG we may assume $a=0$ $(\mathrm{b}+\mathrm{c}=2)$, then : inequality becomes : $\mathrm{b}^2+\mathrm{c}^2 \overset{?}{\geq} 2\mathrm{b}^3\mathrm{c}^3$ and $\because \mathrm{b}^2+\mathrm{c}^2 \overset{A-G}{\geq} 2\mathrm{bc}$ $\therefore$ it suffices to prove : $2\mathrm{bc} \overset{?}{\geq} 2\mathrm{b}^3\mathrm{c}^3 \Leftrightarrow \mathrm{bc} \overset{?}{\leq} 1 \rightarrow$ true $\because 2=\mathrm{b}+\mathrm{c} \overset{A-G}{\geq} 2\sqrt{\mathrm{bc}}$ and so, we now focus on the case when : $a, \mathrm{b}, \mathrm{c}>0$ and assigning $\mathrm{b}+\mathrm{c}=x$, $\mathrm{c}+a=\mathrm{y}, a+\mathrm{b}=\mathrm{z} \Rightarrow x+\mathrm{y}-\mathrm{z}=2\mathrm{c}>0, y+z-x=2a>0$ & $\mathrm{z}+x-\mathrm{y}=2\mathrm{b}>0 \Rightarrow x+\mathrm{y}>z, \mathrm{y}+\mathrm{z}>x, \mathrm{z}+x>y \Rightarrow x, \mathrm{y}, \mathrm{z}$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=\mathrm{s}, \mathrm{R}, \mathrm{r}$ (say); then :

$$\sum_{\text{cyc}} a=\mathrm{s}, abc=\mathrm{r}^2\mathrm{s}, \sum_{\text{cyc}} ab=4\mathrm{Rr}+\mathrm{r}^2, \sum_{\text{cyc}} a^2=\mathrm{s}^2-8\mathrm{Rr}-2\mathrm{r}^2,$$

$$\sum_{\text{cyc}} a^3\mathrm{b}^3=(4\mathrm{Rr}+\mathrm{r}^2)^3-12\mathrm{Rr}^3\mathrm{s}^2 \text{ and then :}$$

$$\sum_{\text{cyc}} a^2 \overset{?}{\geq} 2\left(\sum_{\text{cyc}} a^3\mathrm{b}^3+4a^2\mathrm{b}^2\mathrm{c}^2\right) \overset{a+b+c=2}{\Leftrightarrow}$$

$$\frac{1}{16}\cdot\left(\sum_{\text{cyc}} a^2\right)\left(\sum_{\text{cyc}} a\right)^4 \overset{?}{\geq} 2\left(\sum_{\text{cyc}} a^3\mathrm{b}^3+4a^2\mathrm{b}^2\mathrm{c}^2\right)$$

$$\Leftrightarrow (\mathrm{s}^2-8\mathrm{Rr}-2\mathrm{r}^2)\mathrm{s}^4 \overset{?}{\geq} 32((4\mathrm{Rr}+\mathrm{r}^2)^3-12\mathrm{Rr}^3\mathrm{s}^2+4\mathrm{r}^4\mathrm{s}^2)$$

$$\Leftrightarrow \mathrm{s}^6-(8\mathrm{Rr}+2\mathrm{r}^2)\mathrm{s}^4+\mathrm{r}^3(384\mathrm{R}-128\mathrm{r})\mathrm{s}^2-\mathrm{r}^3(4\mathrm{R}+\mathrm{r})^3 \underset{(*)}{\overset{?}{\geq}} 0$$

and $\because (\mathrm{s}^2-16\mathrm{Rr}+5\mathrm{r}^2)^3 \overset{\text{Gerretsen}}{\geq} 0$ $\therefore$ in order to prove $(*)$, it suffices to prove :

$$\text{LHS of } (*) \overset{?}{\geq} (\mathrm{s}^2-16\mathrm{Rr}+5\mathrm{r}^2)^3 \Leftrightarrow (40\mathrm{R}-17\mathrm{r})\mathrm{s}^4 -$$

$$\mathrm{r}(768\mathrm{R}^2-864\mathrm{Rr}+203\mathrm{r}^2)\mathrm{s}^2+\mathrm{r}^2(2048\mathrm{R}^3-5376\mathrm{R}^2\mathrm{r}+816\mathrm{Rr}^2-157\mathrm{r}^3) \underset{(**)}{\overset{?}{\geq}} 0$$

and $\because (40\mathrm{R}-17\mathrm{r})(\mathrm{s}^2-16\mathrm{Rr}+5\mathrm{r}^2)^2 \overset{\text{Gerretsen}}{\geq} 0$ $\therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**) \overset{?}{\geq} (40\mathrm{R}-17\mathrm{r})(\mathrm{s}^2-16\mathrm{Rr}+5\mathrm{r}^2)^2$

$$\Leftrightarrow (512\mathrm{R}^2-80\mathrm{Rr}-33\mathrm{r}^2)\mathrm{s}^2 \underset{(***)}{\overset{?}{\geq}} \mathrm{r}(8192\mathrm{R}^3-5376\mathrm{R}^2\mathrm{r}+2904\mathrm{Rr}^2-268\mathrm{r}^3)$$

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$$\text{and finally, LHS of } (***) \overset{\text{Gerretsen}}{\geq} (512R^2 - 80Rr - 33r^2)(16Rr - 5r^2) \overset{?}{\geq}$$

$$\text{RHS of } (***) \Leftrightarrow r^2\left((1536R + 40r)(R - 2r) + 513r^2\right) \overset{?}{\geq} 0 \to \text{true}$$

$$\text{(strict inequality)} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true } \forall \, \Delta \, XYZ_{s,R,r}$$

$$\therefore \sum_{cyc} a^2 > 2\left(\sum_{cyc} a^3b^3 + 4a^2b^2c^2\right) \text{ whenever } a, b, c > 0 \text{ and combining all cases,}$$

$$\text{we have : } \sum_{cyc} a^2 \geq 2\left(\sum_{cyc} a^3b^3 + 4a^2b^2c^2\right) \forall \, a, b, c \geq 0 \,\middle|\, a + b + c = 2,$$

$$'' ='' \text{ iff } \begin{pmatrix} a = 0 \\ b = c = 1 \end{pmatrix} \text{ or } \begin{pmatrix} b = 0 \\ c = a = 1 \end{pmatrix} \text{ or } \begin{pmatrix} c = 0 \\ a = b = 1 \end{pmatrix} \text{ (QED)}$$