ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $x^2 + y^2 + z^2 + xyz = 4$ then prove that :

$$\sqrt{2-x} + \sqrt{2-y} + \sqrt{2-z} \geq 2 + \sqrt{(2-x)(2-y)(2-z)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, } (4-y^2), (4-z^2) = (x^2+z^2+xyz), (x^2+y^2+xyz) \overset{x,y,z>0}{>} 0 \text{ and we have : } x^2+y^2+z^2+xyz = 4 \Rightarrow x^2 + x.yz + (y^2+z^2-4) = 0$$

$$\Rightarrow x = \frac{-yz \pm \sqrt{y^2z^2 - 4(y^2+z^2-4)}}{2} = \frac{-yz \pm \sqrt{y^2(z^2-4) - 4(z^2-4)}}{2}$$

$$= \frac{-yz \pm \sqrt{(4-y^2)(4-z^2)}}{2} = \frac{-yz + \sqrt{(4-y^2)(4-z^2)}}{2} \ (\because x > 0)$$

$$\Rightarrow 2 - x = 2 - \frac{-yz + \sqrt{(4-y^2)(4-z^2)}}{2} = \frac{8 + 2yz - 2\sqrt{(4-y^2)(4-z^2)}}{4}$$

$$\overset{A-G}{\geq} \frac{8 + 2yz - (8 - y^2 - z^2)}{4} = \frac{y^2+z^2+2yz}{4} = \frac{(y+z)^2}{4}$$

$$\Rightarrow \sqrt{2-x} \geq \frac{y+z}{2} \text{ and analogs} \Rightarrow \sum_{zyz} \sqrt{2-x} \overset{①}{\geq} \sum_{zyz} \frac{y+z}{2} = \sum_{cyc} x$$

$$\Rightarrow \sum_{cyc} x \leq \sum_{cyc} \sqrt{2-x} \overset{CBS}{\leq} \sqrt{3}.\sqrt{6 - \sum_{cyc} x} \Rightarrow t^2 \leq 18 - 3t \left(t = \sum_{cyc} x\right)$$

$$\Rightarrow (t-3)(t+6) \leq 0 \Rightarrow t = \sum_{cyc} x \overset{②}{\leq} 3 \text{ and we have : } \left(\sum_{cyc} x\right)^2 - 4 =$$

$$\sum_{cyc} x^2 + 2\sum_{cyc} xy - \left(\sum_{cyc} x^2 + xyz\right) = 2xyz \sum_{cyc} \frac{1}{x} - xyz \overset{Bergstrom}{\geq} \frac{18xyz}{\sum_{cyc} x} - xyz$$

$$\overset{\text{via } ②}{\geq} 6xyz - xyz = 5xyz > 0 \Rightarrow \left(\sum_{cyc} x\right)^2 > 4 \Rightarrow \sum_{cyc} x \overset{③}{\geq} 2$$

$$\therefore \boxed{\sum_{zyz} \sqrt{2-x} - 2 - \sqrt{(2-x)(2-y)(2-z)}} \overset{\text{via } ①}{\geq}$$

$$\sum_{cyc} x - 2 - \sqrt{8 + 2\sum_{cyc} xy - 4\sum_{cyc} x + \sum_{cyc} x^2 - 4} \left(\because -xyz = \sum_{cyc} x^2 - 4\right)$$

$$= \sum_{cyc} x - 2 - \sqrt{4 - 4\sum_{cyc} x + \left(\sum_{cyc} x\right)^2} = \sum_{cyc} x - 2 - \sqrt{\left(\sum_{cyc} x - 2\right)^2}$$

$$\overset{\text{via} ③}{=} \sum_{cyc} x - 2 - \left(\sum_{cyc} x - 2\right) = 0 \text{ and so,}$$

$$\sqrt{2-x} + \sqrt{2-y} + \sqrt{2-z} \geq 2 + \sqrt{(2-x)(2-y)(2-z)}$$

$$\forall\, x, y, z > 0 \,\Big|\, x^2 + y^2 + z^2 + xyz = 4, '' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$