ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c\ge 0$ and $a+b+c=3$ then prove that :

$$\frac{5ab}{3c+ab}+\frac{3bc}{3a+bc}+\frac{3ca}{3b+ca}\le 5$$

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$$3c+ab\overset{a+b+c=3}{=}\frac{3c+a(3-c-a)}{2}+\frac{3c+b(3-b-c)}{2}$$

$$=\frac{(3-a)(c+a)}{2}+\frac{(3-b)(b+c)}{2}\overset{a+b+c=3}{=}\frac{(b+c)(c+a)}{2}+\frac{(c+a)(b+c)}{2}$$

$$\Rightarrow 3c+ab\overset{①}{=}(b+c)(c+a)$$

Also, $3a+bc\overset{a+b+c=3}{=}3a+c(3-c-a)=(3-c)(c+a)\overset{a+b+c=3}{=}(a+b)(c+a)$

$$\Rightarrow 3a+bc\overset{②}{=}(a+b)(c+a)$$

And, $3b+ca\overset{a+b+c=3}{=}3b+c(3-b-c)=(3-c)(b+c)\overset{a+b+c=3}{=}(a+b)(b+c)$

$$\Rightarrow 3b+ca\overset{③}{=}(a+b)(b+c)$$

Now, $\frac{5ab}{3c+ab}+\frac{3bc}{3a+bc}+\frac{3ca}{3b+ca}\overset{?}{\le}5\Leftrightarrow\frac{3bc}{3a+bc}+\frac{3ca}{3b+ca}\overset{?}{\le}5-\frac{5ab}{3c+ab}$

$$=\frac{15c}{3c+ab}\Leftrightarrow\frac{5}{3c+ab}\overset{?}{\ge}\frac{a}{3b+ca}+\frac{b}{3a+bc}\ (\because c\ge 0)$$

$$\overset{\text{via }①,②\text{ and }③}{\Leftrightarrow}\frac{5}{(b+c)(c+a)}\overset{?}{\ge}\frac{a}{(a+b)(b+c)}+\frac{b}{(a+b)(c+a)}$$

$$\Leftrightarrow 5(a+b)\overset{?}{\ge}a(c+a)+b(b+c)=c(a+b)+a^2+b^2$$

$$\Leftrightarrow(3-c+2)(a+b)\overset{?}{\ge}a^2+b^2\overset{a+b+c=3}{\Leftrightarrow}(a+b+2)(a+b)\overset{?}{\ge}a^2+b^2$$

$\Leftrightarrow 2ab+2(a+b)\overset{?}{\ge}0\to$ true $\because a,b\ge 0$ and this last inequality is a strict one since a,b cannot be simultaneously equal to zero as that would make $\frac{3bc}{3a+bc}$ and $\frac{3ca}{3b+ca}$ undefined $\therefore\frac{5ab}{3c+ab}+\frac{3bc}{3a+bc}+\frac{3ca}{3b+ca}\le 5$

$$\forall a,b,c\ge 0\,\middle|\,a+b+c=3,\text{''}=\text{''}\text{ iff}\begin{pmatrix}c=0\\a=k\\b=3-k\\\text{with }0<k<3\end{pmatrix}\text{(QED)}$$