ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $x^2 + y^2 + z^2 \leq 3$ then prove that :

$$\sqrt{\frac{x}{(2-y)(2-z)}} + \sqrt[3]{\frac{y}{(2-z)(2-x)}} + \sqrt[4]{\frac{z}{(2-x)(2-y)}} \leq 3$$

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$$x^2 + y^2 + z^2 \leq 3 \Rightarrow x^2 \leq 3 - y^2 - z^2 \text{ and}$$
$$\because x^2 > 0 \therefore 3 - y^2 - z^2 \geq x^2 > 0 \text{ and so}$$
$$x \leq \sqrt{3 - y^2 - z^2} = \sqrt{(3 - y^2 - z^2).1} \overset{\text{AM-GM}}{\leq} \frac{3 - y^2 - z^2 + 1}{2}$$
$$\overset{?}{\leq} (2-y)(2-z) \Leftrightarrow 8 - 4(y+z) + 2yz \overset{?}{\geq} 4 - y^2 - z^2$$
$$\Leftrightarrow y^2 + z^2 + 2yz - 4(y+z) + 4 \overset{?}{\geq} 0 \Leftrightarrow (y+z)^2 - 4(y+z) + 4 \overset{?}{\geq} 0$$
$$\Leftrightarrow (y+z-2)^2 \overset{?}{\geq} 0 \to \text{true} \Rightarrow x \leq (2-y)(2-z) \Rightarrow \sqrt{\frac{x}{(2-y)(2-z)}} \overset{①}{\leq} 1 \text{ and}$$
$$\text{analogously}, y \leq (2-z)(2-x) \Rightarrow \sqrt[3]{\frac{y}{(2-z)(2-x)}} \overset{②}{\leq} 1 \text{ and also},$$
$$z \leq (2-x)(2-y) \Rightarrow \sqrt[4]{\frac{z}{(2-x)(2-y)}} \overset{③}{\leq} 1 \therefore ① + ② + ③ \Rightarrow$$
$$\sqrt{\frac{x}{(2-y)(2-z)}} + \sqrt[3]{\frac{y}{(2-z)(2-x)}} + \sqrt[4]{\frac{z}{(2-x)(2-y)}} \leq 3 \ \forall\, x, y, z > 0,$$
$$'' ='' \text{ iff } y + z = 2 \wedge 3 - y^2 - z^2 = 1 \wedge x^2 = 3 - y^2 - z^2$$
$$\Rightarrow '' ='' \text{ iff } y + z = 2 \wedge yz = 1 \wedge x = 1 \Rightarrow '' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$