

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, a + b + c = 3$ then:

$$\sum_{cyc} \frac{a^2}{\sqrt{6a^2 + 13a + 6}} \geq \frac{3}{5}$$

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Lemma:

$$\forall x > 0, \sqrt{6x^2 + 13x + 6} \leq \frac{5}{2}(x + 1)$$

Proof: we need to show

$$\sqrt{6x^2 + 13x + 6} \leq \frac{5}{2}(x + 1)$$

$$25(x + 1)^2 \geq 4(6x^2 + 13x + 6) \text{ or } (x - 1)^2 \geq 0 \text{ true}$$

$$\sum_{cyc} \frac{a^2}{\sqrt{6a^2 + 13a + 6}} \stackrel{\text{lemma}}{\geq} \sum_{cyc} \frac{a^2}{\frac{5}{2}(a + 1)} \stackrel{\text{Bergstrom}}{\geq} \frac{2}{5} \frac{(a + b + c)^2}{(a + b + c + 3)} \stackrel{a+b+c=3}{=} \frac{3}{5}$$

Equality holds for $a=b=c=1$.