ROMANIAN MATHEMATICAL MAGAZINE

If $a,\mathrm{b},\mathrm{c}>0$ and $a+\mathrm{b}+\mathrm{c}\geq 3$ then prove that :

$$\frac{1}{4a^4+b^4+c^4}+\frac{1}{4b^4+c^4+a^4}+\frac{1}{4c^4+a^4+b^4}\leq\frac{1}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}a^2\geq\frac{1}{3}\left(\sum_{cyc}a\right)^2\overset{a+b+c\geq 3}{\geq}3\Rightarrow 1\leq\frac{1}{9}\left(\sum_{cyc}a^2\right)^2\Rightarrow\sum_{cyc}\frac{1}{4a^4+b^4+c^4}\leq$$

$$\frac{1}{9}\left(\sum_{cyc}a^2\right)^2\cdot\left(\sum_{cyc}\frac{1}{4a^4+b^4+c^4}\right)=\frac{1}{9}\left(\sum_{cyc}x\right)^2\cdot\left(\sum_{cyc}\frac{1}{4x^2+y^2+z^2}\right)\begin{pmatrix}x=a^2,\\ y=b^2,\\ z=c^2\end{pmatrix}$$

$$=\frac{1}{9}\left(\sum_{cyc}x\right)^2\cdot\sum_{cyc}\frac{(4y^2+z^2+x^2)(4z^2+x^2+y^2)}{(4x^2+y^2+z^2)(4y^2+z^2+x^2)(4z^2+x^2+y^2)}\overset{?}{\leq}\frac{1}{2}$$

$$\Leftrightarrow 2\sum_{cyc}x^6+13\sum_{cyc}x^4y^2+13\sum_{cyc}x^2y^4+60x^2y^2z^2\overset{?}{\geq}4\sum_{cyc}x^5y+4\sum_{cyc}xy^5+$$

$$4xyz\sum_{cyc}x^3+12\sum_{cyc}x^3y^3+12xyz\left(\sum_{cyc}x^2y+\sum_{cyc}xy^2\right)\Leftrightarrow 2\sum_{cyc}x^6+$$

$$13\sum_{cyc}\left(x^2y^2\left(\sum_{cyc}x^2-z^2\right)\right)+60x^2y^2z^2\overset{?}{\geq}4\sum_{cyc}\left(xy\left(\sum_{cyc}x^4-z^4\right)\right)+$$

$$4xyz\sum_{cyc}x^3+12\sum_{cyc}x^3y^3+12xyz\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-3xyz\right)$$

$$\Leftrightarrow 2\sum_{cyc}x^6+13\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x^2y^2\right)+57x^2y^2z^2\boxed{\overset{?}{\underset{(*)}{\geq}}}4\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^4\right)+$$

$$12\sum_{cyc}x^3y^3+12xyz\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)$$ & *assigning* $y+z=A,z+x=B,$

$x+y=C\Rightarrow A+B-C=2z>0,B+C-A=2x>0$ and $C+A-B=2y>0$

$\Rightarrow A+B>C,B+C>A,C+A>B\Rightarrow A,B,C$ *form sides of a* triangle

with semiperimeter, circumradius and inradius $=s,R,r$ (*say*) and then :

$$2\sum_{cyc}x=\sum_{cyc}A=2s\Rightarrow\sum_{cyc}x=s;\ also\ x=s-A,y=s-B,z=s-C\Rightarrow xyz=r^2s,$$

ROMANIAN MATHEMATICAL MAGAZINE

$\sum_{cyc} xy = 4Rr + r^2, \sum_{cyc} x^2 = s^2 - 8Rr - 2r^2, \sum_{cyc} x^2y^2 = r^2((4R+r)^2 - 2s^2), \sum_{cyc} x^4$
$= (s^2 - 8Rr - 2r^2)^2 - 2r^2((4R+r)^2 - 2s^2), \sum_{cyc} x^3y^3 = (4Rr + r^2)^3 - 12Rr^3s^2$
and $\sum_{cyc} x^6 = (s^3 - 12Rrs)^2 - 2((4Rr + r^2)^3 - 12Rr^3s^2)$ & via such substitutions and following simplification, $(*)$ becomes :

$$s^6 - (32Rr + 15r^2)s^4 + r^2(376R^2 + 260Rr + 55r^2)s^2 - 25r^3(4R+r)^3 \underset{(**)}{\overset{?}{\geq}} 0$$

Since $(s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(**)$, it suffices to prove :
LHS of $(**) \overset{?}{\geq} (s^2 - 16Rr + 5r^2)^3 \Leftrightarrow (8R - 15r)s^4 - r(196R^2 - 370Rr + 10r^2)s^2$
$+r^2(1248R^3 - 2520R^2r + 450Rr^2 - 75r^3) \underset{(***)}{\overset{?}{\geq}} 0$ and again, since
$(8R - 15r)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(***)$,
it suffices to prove : LHS of $(***) \overset{?}{\geq} (8R - 15r)(s^2 - 16Rr + 5r^2)^2$
$\Leftrightarrow (6R^2 - 19Rr + 14r^2)s^2 \underset{(****)}{\overset{?}{\geq}} r(80R^3 - 260R^2r + 215Rr^2 - 30r^3)$ and
finally, $(6R^2 - 19Rr + 14r^2)s^2 \overset{\text{Gerretsen}}{\geq} (6R^2 - 19Rr + 14r^2)(16Rr - 5r^2) \overset{?}{\geq}$
$r(80R^3 - 260R^2r + 215Rr^2 - 30r^3) \Leftrightarrow 2r(R - 2r)^2(8R - 5r) \overset{?}{\geq} 0 \to$ true
$\because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*)$ is true and so,

$$\frac{1}{4a^4 + b^4 + c^4} + \frac{1}{4b^4 + c^4 + a^4} + \frac{1}{4c^4 + a^4 + b^4} \leq \frac{1}{2} \ \forall\, a, b, c > 0 \mid a + b + c \geq 3,$$

$'' = ''$ iff $a = b = c = 1$ (QED)