ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ and $a + b = 4$ then prove that :

$$\frac{a^2}{\sqrt{a^3+8}} + \frac{b^2}{\sqrt{b^3+8}} \le 2$$

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$$\frac{a^2}{\sqrt{a^3+8}} + \frac{b^2}{\sqrt{b^3+8}} = \sqrt{a}.\frac{a.\sqrt{a}}{\sqrt{a^3+8}} + \sqrt{b}.\frac{b.\sqrt{b}}{\sqrt{b^3+8}} \overset{\text{CBS}}{\le}$$

$$\sqrt{a+b}.\sqrt{\frac{a^3}{a^3+8} + \frac{b^3}{b^3+8}} \overset{a+b=4}{\le} 2.\sqrt{\frac{a^3+8-8}{a^3+8} + \frac{b^3+8-8}{b^3+8}}$$

$$= 2.\sqrt{2 - 8\left(\frac{1}{a^3+8} + \frac{2}{b^3+8}\right)} = 2.\sqrt{2 - \frac{8(a^3+b^3+16)}{a^3b^3+8(a^3+b^3)+64}}$$

$$\le 2.\sqrt{2 - \frac{8(a^3+b^3+16)}{64+8(a^3+b^3)+64}} \left(\because 4 = a+b \overset{\text{AM-GM}}{\ge} 2.\sqrt{ab} \Rightarrow ab \le 4\right)$$

$$= 2.\sqrt{2 - \frac{8(a^3+b^3+16)}{8(a^3+b^3+16)}} = 2 \text{ and so, } \frac{a^2}{\sqrt{a^3+8}} + \frac{b^2}{\sqrt{b^3+8}} \le 2$$

$$\forall\, a, b > 0 \,\middle|\, a+b=4, '' = '' \text{ iff } a = b = 2 \text{ (QED)}$$