ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b} > 0$ and $a^3 + \mathrm{b}^3 = 2$ then prove that :

$$\frac{a^3}{\sqrt[3]{a+7}} + \frac{\mathrm{b}^3}{\sqrt[3]{\mathrm{b}+7}} \le 1$$

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$$\frac{a^3}{\sqrt[3]{a+7}} \overset{?}{\le} \frac{71a^3+1}{144} \Leftrightarrow (a+7)(71a^3+1)^3 \overset{?}{\ge} (144)^3 a^9$$

$$\Leftrightarrow (a-1)^2 \begin{pmatrix} 23a^8 + 15a^7 + 7a^6 + 64a^6(5592a^2 + 3675a + 1758) + \\ 4946a^5 + 3234a^4 + 1522a^3 + 23a^2 + 15a + 7 \end{pmatrix} \overset{?}{\ge} 0$$

$$\to \text{true} \because a > 0 \therefore \frac{a^3}{\sqrt[3]{a+7}} \le \frac{71a^3+1}{144} \text{ and } \frac{\mathrm{b}^3}{\sqrt[3]{\mathrm{b}+7}} \le \frac{71\mathrm{b}^3+1}{144} \text{ and so,}$$

$$\frac{a^3}{\sqrt[3]{a+7}} + \frac{\mathrm{b}^3}{\sqrt[3]{\mathrm{b}+7}} \le \frac{71(a^3+\mathrm{b}^3)+2}{144} \overset{a^3+\mathrm{b}^3=2}{=} \frac{142+2}{144} = 1$$

$$\therefore \frac{a^3}{\sqrt[3]{a+7}} + \frac{\mathrm{b}^3}{\sqrt[3]{\mathrm{b}+7}} \le 1 \ \forall\, a, \mathrm{b} > 0 \,\Big|\, a^3 + \mathrm{b}^3 = 2, '' = '' \text{ iff } a = \mathrm{b} = 1 \text{ (QED)}$$