ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ *then prove that*:

$$\sum_{cyc} \frac{c^3 - a^3}{\sqrt{(1+a^2)(1+b^2)}} \ge 0$$

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Case 1 $a \ge b \ge c$ or $a \ge c \ge b$ or $b \ge c \ge a$ or $c \ge b \ge a$ *and then*:

$$\sum_{cyc} \frac{c^3 - a^3}{\sqrt{(1+a^2)(1+b^2)}} =$$

$$\frac{c^3 - b^3 + b^3 - a^3}{\sqrt{(1+a^2)(1+b^2)}} + \frac{a^3 - b^3}{\sqrt{(1+b^2)(1+c^2)}} + \frac{b^3 - c^3}{\sqrt{(1+c^2)(1+a^2)}}$$

$$= \frac{b^3 - c^3}{\sqrt{1+a^2}} \cdot \left(\frac{1}{\sqrt{1+c^2}} - \frac{1}{\sqrt{1+b^2}}\right) + \frac{a^3 - b^3}{\sqrt{1+b^2}} \cdot \left(\frac{1}{\sqrt{1+c^2}} - \frac{1}{\sqrt{1+a^2}}\right)$$

$$= \frac{b^3 - c^3}{\sqrt{(1+a^2)(1+b^2)(1+c^2)}} \cdot \frac{1 + b^2 - 1 - c^2}{\sqrt{1+b^2} + \sqrt{1+c^2}} +$$

$$\frac{a^3 - b^3}{\sqrt{(1+a^2)(1+b^2)(1+c^2)}} \cdot \frac{1 + a^2 - 1 - c^2}{\sqrt{1+c^2} + \sqrt{1+a^2}}$$

$$= \frac{(b-c)^2(b^2 + bc + c^2)(b+c)}{\sqrt{(1+a^2)(1+b^2)(1+c^2)} \cdot \sqrt{(1+b^2)(1+c^2)}} +$$

$$\frac{(a-b)(a-c)(a^2 + ab + b^2)(a+c)}{\sqrt{(1+a^2)(1+b^2)(1+c^2)} \cdot \sqrt{(1+c^2)(1+a^2)}} \ge 0$$

$(\because a \ge b \ge c \text{ or } a \ge c \ge b \text{ or } b \ge c \ge a \text{ or } c \ge b \ge a)$

Case 2 $b \ge a \ge c$ or $c \ge a \ge b$ *and then*: $\sum_{cyc} \frac{c^3 - a^3}{\sqrt{(1+a^2)(1+b^2)}} =$

$$\frac{c^3 - a^3}{\sqrt{(1+a^2)(1+b^2)}} + \frac{a^3 - c^3 + c^3 - b^3}{\sqrt{(1+b^2)(1+c^2)}} + \frac{b^3 - c^3}{\sqrt{(1+c^2)(1+a^2)}}$$

$$= \frac{c^3 - a^3}{\sqrt{1+b^2}} \cdot \left(\frac{1}{\sqrt{1+a^2}} - \frac{1}{\sqrt{1+c^2}}\right) + \frac{b^3 - c^3}{\sqrt{1+c^2}} \cdot \left(\frac{1}{\sqrt{1+a^2}} - \frac{1}{\sqrt{1+b^2}}\right)$$

$$= \frac{c^3 - a^3}{\sqrt{(1+a^2)(1+b^2)(1+c^2)}} \cdot \frac{1 + c^2 - 1 - a^2}{\sqrt{1+c^2} + \sqrt{1+a^2}} +$$

$$\frac{b^3 - c^3}{\sqrt{(1+a^2)(1+b^2)(1+c^2)}} \cdot \frac{1 + b^2 - 1 - a^2}{\sqrt{1+b^2} + \sqrt{1+a^2}}$$

$$= \frac{(c-a)^2(c^2 + ca + a^2)(c+a)}{\sqrt{(1+a^2)(1+b^2)(1+c^2)} \cdot \sqrt{(1+c^2)(1+a^2)}} +$$

$$\frac{(b-c)(b-a)(b^2+bc+c^2)(a+b)}{\sqrt{(1+a^2)(1+b^2)(1+c^2)}.\sqrt{(1+a^2)(1+b^2)}} \geq 0 \; (\because b \geq a \geq c \text{ or } c \geq a \geq b)$$

$$\text{and so, combining both cases,} \sum_{cyc} \frac{c^3-a^3}{\sqrt{(1+a^2)(1+b^2)}} \geq 0 \; \forall\, a, b, c > 0,$$

$$'' = '' \text{ iff } a = b = c \text{ (QED)}$$