ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z \geq 0$ and $x + y + z = 1$ then prove that :

$$\frac{1}{x^3+y^3+z^3}+\frac{3}{xyz}\geq 90$$

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Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b$ $\Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) $\Rightarrow \sum_{cyc} x = s, xyz = r^2 s$ and $\sum_{cyc} x^3 = s^3 - 12Rrs$

$$\therefore \frac{1}{x^3+y^3+z^3}+\frac{3}{xyz}\overset{?}{\geq} 90 \Leftrightarrow \frac{\left(\sum_{cyc} x\right)^3}{\sum_{cyc} x^3}+\frac{3}{xyz}\left(\sum_{cyc} x\right)^3 \overset{?}{\geq} 90$$

$$\Leftrightarrow xyz\left(\sum_{cyc} x\right)^3 + 3\left(\sum_{cyc} x^3\right)\left(\sum_{cyc} x\right)^3 \overset{?}{\geq} 90xyz\left(\sum_{cyc} x^3\right)$$

$$\Leftrightarrow r^2s.s^3 + 3s^3(s^3-12Rrs) \overset{?}{\geq} 90r^2s(s^3-12Rrs)$$

$$\Leftrightarrow 3s^4 - (36Rr+89r^2)s^2 + 1080Rr^3 \underset{(*)}{\overset{?}{\geq}} 0$$

Now, since $(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(*)$,

it suffices to prove : LHS of $(*) \overset{?}{\geq} 3(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (60R-119r)s^2 \underset{(**)}{\overset{?}{\geq}} r(768R^2-1560Rr+75r^2) \text{ and finally,}$$

$$(60R-119r)(16Rr-5r^2) \overset{?}{\geq} r(768R^2-1560Rr+75r^2)$$

$$\Leftrightarrow 4r(R-2r)(48R-65r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{1}{x^3+y^3+z^3}+\frac{3}{xyz} \geq 90 \;\forall\, x,y,z>0 \,\big|\, x+y+z=1,$$

$$'' ='' \text{ iff } x=y=z=\frac{1}{3} \text{ (QED)}$$