ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0$ *then prove that* :

$$\frac{(xy)^3}{z^9+1}+\frac{(yz)^3}{x^9+1}+\frac{(zx)^3}{y^9+1}\geq\frac{3(xyz)^2}{(xyz)^3+1}$$

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$$\text{Let } f(t)=\frac{1}{t^{11}+t^2}\ \forall\ t>0\ \text{and then}: f''(t)=\frac{6(22t^{18}-4t^9+1)}{t^4(t^9+1)^3}$$

$$=\frac{6(18t^{18}+(2t^9-1)^2)}{t^4(t^9+1)^3}>0\ \therefore f(t)\ \text{is convex and so,}$$

$$\text{via Weighted Jensen's Inequality,}\frac{(xy)^3}{z^9+1}+\frac{(yz)^3}{x^9+1}+\frac{(zx)^3}{y^9+1}=(xyz)^2.\sum_{cyc}\frac{yz}{x^{11}+x^2}$$

$$\geq(xyz)^2.\left(\sum_{cyc}yz\right).\frac{1}{\left(\frac{\sum_{cyc}(yz.x)}{\sum_{cyc}yz}\right)^{11}+\left(\frac{\sum_{cyc}(yz.x)}{\sum_{cyc}yz}\right)^{2}}\overset{AM-GM}{\geq}$$

$$(xyz)^2.\left(3.(xyz)^{\frac{2}{3}}\right).\frac{1}{\left(\frac{3xyz}{3.(xyz)^{\frac{2}{3}}}\right)^{11}+\left(\frac{3xyz}{3.(xyz)^{\frac{2}{3}}}\right)^{2}}=\frac{3(xyz)^2.(xyz)^{\frac{2}{3}}}{(xyz)^{\frac{11}{3}}+(xyz)^{\frac{2}{3}}}=\frac{3(xyz)^2}{(xyz)^3+1}$$

$$\text{and so,}\frac{(xy)^3}{z^9+1}+\frac{(yz)^3}{x^9+1}+\frac{(zx)^3}{y^9+1}\geq\frac{3(xyz)^2}{(xyz)^3+1}\ \forall\ x,y,z>0,$$

$$''='' \text{ iff } x=y=z\ (\text{QED})$$