ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0, a + b = 2$ then:

$$\frac{a}{b^2 + 1} + \frac{b}{a^2 + 1} \geq 1$$

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Denote:

$$\begin{cases} S = x + y \\ P = xy \end{cases}$$

$$2 = a + b \overset{AM-GM}{\geq} 2\sqrt{ab} = 2\sqrt{P} \Rightarrow 2 \geq 2\sqrt{P} \Rightarrow P \leq 1$$

$$P - 1 \leq 0 \quad (1)$$

$$P > 0 \Rightarrow P + 5 > 0 \quad (2)$$

$$\frac{a}{b^2 + 1} + \frac{b}{a^2 + 1} = \frac{a(a^2 + 1) + b(b^2 + 1)}{(a^2 + 1)(b^2 + 1)} =$$

$$= \frac{a^3 + b^3 + a + b}{a^2 b^2 + a^2 + b^2 + 1} = \frac{S^3 - 3SP + S}{P^2 + S^2 - 2P + 1} = \frac{8 - 6P + 2}{P^2 + 4 - 2P + 1} = \frac{10 - 6P}{P^2 - 2P + 5} \geq 1 \Leftrightarrow$$

$$10 - 6P \geq P^2 - 2P + 5$$

$$P^2 + 4P - 5 \leq 0$$

By $(1), (2)$:

$$(P - 1)(P + 5) \leq 0$$

Equality holds for $a = b = 1$.