ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c \geq 0$ and $a+b+c=1$ then prove that :

$$\frac{16}{27} \leq (1-5ab)^2+(1-5bc)^2+(1-5ca)^2 \leq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Case 1 Exactly 2 variables $=0$ and WLOG we may assume $b=c=0$

$(a=1)$ and then : $\sum_{cyc}(1-5bc)^2 = 3$ and $\because \frac{16}{27} < 3 = 3 \therefore \frac{16}{27} < \sum_{cyc}(1-5bc)^2 = 3$

Case 2 Exactly 1 variable $=0$ and WLOG we may assume $a=0$ $\binom{b+c=1;}{b,c>0}$

and then : $\sum_{cyc}(1-5bc)^2 = 2+(1-5bc)^2 \overset{?}{<} 3 \Leftrightarrow 1-10bc+25b^2c^2 \overset{?}{<} 1$

$\Leftrightarrow 2(b+c)^2 \overset{?}{>} 5bc \ (\because b+c=1) \Leftrightarrow 2(b-c)^2+3bc \overset{?}{>} 0 \to$ true $\because b,c>0$

$\therefore \sum_{cyc}(1-5bc)^2 < 3$ and $\because 2+(1-5bc)^2 > \frac{16}{27} \therefore \sum_{cyc}(1-5bc)^2 > \frac{16}{27}$

Case 3 $a,b,c>0$ and then : $\sum_{cyc}(1-5bc)^2 = 3-10\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2 +$

$25\sum_{cyc}a^2b^2 \ (\because a+b+c=1) \leq 3-30\left(\sum_{cyc}ab\right)^2+25\sum_{cyc}a^2b^2$

$= 3-5\sum_{cyc}a^2b^2-60abc\left(\sum_{cyc}a\right) < 3 \ (\because a,b,c>0) \therefore \boxed{\sum_{cyc}(1-5bc)^2 < 3}$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s,R,r$ (say);

then : $\sum_{cyc}a = s, abc = r^2s, \sum_{cyc}ab = 4Rr+r^2, \sum_{cyc}a^2b^2 = r^2((4R+r)^2-2s^2)$

and then, $\sum_{cyc}(1-5bc)^2 \overset{?}{\geq} \frac{16}{27}$ becomes :

$$3\left(\sum_{cyc}a\right)^4-10\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^2+25\sum_{cyc}a^2b^2 \overset{?}{\geq} \frac{16}{27}.\left(\sum_{cyc}a\right)^4$$

$$\Leftrightarrow 65s^4 - 270(4Rr + r^2)s^2 + 675r^2((4R + r)^2 - 2s^2) \boxed{\overset{?}{\underset{(*)}{\geq}}} 0 \text{ and } \because P =$$

$$65(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) \overset{?}{\geq} P \Leftrightarrow (100R - 227r)s^2 \boxed{\overset{?}{\underset{(**)}{\geq}}} r(584R^2 - 1580Rr + 95r^2)$$

$\boxed{\textit{Case (i)}}$ $100R - 227r \geq 0$ *and then* : LHS of $(**) \overset{\text{Gerretsen}}{\geq}$

$$(100R - 227r)(16Rr - 5r^2) \overset{?}{\geq} \text{RHS of } (**) \Leftrightarrow 8r(R - 2r)(172R - 65r) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (**) \text{ is true}$$

$\boxed{\textit{Case (ii)}}$ $100R - 227r < 0$ *and then* : LHS of $(**) \overset{\text{Gerretsen}}{\geq}$

$$(100R - 227r)(4R^2 + 4Rr + 3r^2) \overset{?}{\geq} \text{RHS of } (**)$$

$$\Leftrightarrow 100t^3 - 273t^2 + 243t - 194 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t - 2)(100t^2 - 73t + 97) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \text{ is true} \therefore \text{combining both cases}, (**) \Rightarrow (*) \text{ is true}$$

$$\forall \Delta XYZ_{s,R,r} \therefore \boxed{\sum_{\text{cyc}} (1 - 5bc)^2 \geq \frac{16}{27}} \therefore \text{combining cases } ①, ② \text{ and } ③,$$

$$\frac{16}{27} \leq \sum_{\text{cyc}} (1 - 5bc)^2 \leq 3, \text{"="} \text{ for lower bound iff } a = b = c = \frac{1}{3} \text{ and}$$

$$\text{"="} \text{ for upper bound iff } \begin{pmatrix} a = 1 \\ b = c = 0 \end{pmatrix} \text{ or } \begin{pmatrix} b = 1 \\ a = c = 0 \end{pmatrix} \text{ or } \begin{pmatrix} c = 1 \\ a = b = 0 \end{pmatrix} \text{ (QED)}$$