ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then prove that:

$$\frac{a-b}{\sqrt{b+1}}+\frac{b-c}{\sqrt{c+1}}+\frac{c-a}{\sqrt{a+1}} \geq 0$$

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**Case 1** $a \geq b \geq c$ or $a \geq c \geq b$ or $b \geq c \geq a$ or $c \geq b \geq a$ and then:

$$\frac{a-b}{\sqrt{b+1}}+\frac{b-c}{\sqrt{c+1}}+\frac{c-a}{\sqrt{a+1}}=\frac{a-c+c-b}{\sqrt{b+1}}+\frac{b-c}{\sqrt{c+1}}+\frac{c-a}{\sqrt{a+1}}$$

$$=(c-a)\left(\frac{1}{\sqrt{a+1}}-\frac{1}{\sqrt{b+1}}\right)+(b-c)\left(\frac{1}{\sqrt{c+1}}-\frac{1}{\sqrt{b+1}}\right)$$

$$=\frac{(c-a)(b-a)}{\sqrt{(a+1)(b+1)}.\left(\sqrt{b+1}+\sqrt{a+1}\right)}+\frac{(b-c)^2}{\sqrt{(c+1)(b+1)}.\left(\sqrt{b+1}+\sqrt{c+1}\right)} \geq 0$$

**Case 2** $b \geq a \geq c$ or $c \geq a \geq b$ and then: $\frac{a-b}{\sqrt{b+1}}+\frac{b-c}{\sqrt{c+1}}+\frac{c-a}{\sqrt{a+1}}$

$$=\frac{a-b}{\sqrt{b+1}}+\frac{b-a+a-c}{\sqrt{c+1}}+\frac{c-a}{\sqrt{a+1}}$$

$$=(a-b)\left(\frac{1}{\sqrt{b+1}}-\frac{1}{\sqrt{c+1}}\right)+(c-a)\left(\frac{1}{\sqrt{a+1}}-\frac{1}{\sqrt{c+1}}\right)$$

$$=\frac{(a-b)(c-b)}{\sqrt{(b+1)(c+1)}.\left(\sqrt{b+1}+\sqrt{c+1}\right)}+\frac{(c-a)^2}{\sqrt{(a+1)(c+1)}.\left(\sqrt{a+1}+\sqrt{c+1}\right)} \geq 0$$

Since we have considered all possible cases concerning the ordering of $a, b, c$ amongst each other, hence we conclude: $\frac{a-b}{\sqrt{b+1}}+\frac{b-c}{\sqrt{c+1}}+\frac{c-a}{\sqrt{a+1}} \geq 0$

$\forall\, a, b, c > 0$, " = " iff $a = b = c$