ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ *then prove that* :

$$\frac{a^2-b^2}{b^3+1}+\frac{b^2-c^2}{c^3+1}+\frac{c^2-a^2}{a^3+1}\geq 0$$

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Case 1 $a\geq b\geq c$ or $a\geq c\geq b$ or $b\geq c\geq a$ or $c\geq b\geq a$ and then :

$$\frac{a^2-b^2}{b^3+1}+\frac{b^2-c^2}{c^3+1}+\frac{c^2-a^2}{a^3+1}=\frac{a^2-c^2+c^2-b^2}{b^3+1}+\frac{b^2-c^2}{c^3+1}+\frac{c^2-a^2}{a^3+1}$$

$$=(c^2-a^2)\left(\frac{1}{a^3+1}-\frac{1}{b^3+1}\right)+(b^2-c^2)\left(\frac{1}{c^3+1}-\frac{1}{b^3+1}\right)$$

$$=\frac{(c-a)(b-a)(c+a)(a^2+ab+b^2)}{(a^3+1)(b^3+1)}+\frac{(b-c)^2(b+c)(b^2+bc+c^2)}{(c^3+1)(b^3+1)}\geq 0$$

Case 2 $b\geq a\geq c$ or $c\geq a\geq b$ and then : $\frac{a^2-b^2}{b^3+1}+\frac{b^2-c^2}{c^3+1}+\frac{c^2-a^2}{a^3+1}$

$$=\frac{a^2-b^2}{b^3+1}+\frac{b^2-a^2+a^2-c^2}{c^3+1}+\frac{c^2-a^2}{a^3+1}$$

$$=(a^2-b^2)\left(\frac{1}{b^3+1}-\frac{1}{c^3+1}\right)+(c^2-a^2)\left(\frac{1}{a^3+1}-\frac{1}{c^3+1}\right)$$

$$=\frac{(a-b)(c-b)(a+b)(b^2+bc+c^2)}{(b^3+1)(c^3+1)}+\frac{(c-a)^2(c+a)(c^2+ca+a^2)}{(a^3+1)(c^3+1)}\geq 0$$

Since we have considered *all* possible cases concerning the ordering of a, b, c amongst each other, hence we conclude : $\frac{a^2-b^2}{b^3+1}+\frac{b^2-c^2}{c^3+1}+\frac{c^2-a^2}{a^3+1}\geq 0$ $\forall\, a, b, c > 0$, " = " iff $a=b=c$