ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then prove that:

$$\frac{a-b}{a^2+1}+\frac{b-c}{b^2+1}+\frac{c-a}{c^2+1}\le 0$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$\boxed{\text{Case 1}}$ $a \ge c \ge b$ or $b \ge c \ge a$ or $b \ge a \ge c$ or $c \ge a \ge b$ and then:

$$\frac{a-b}{a^2+1}+\frac{b-c}{b^2+1}+\frac{c-a}{c^2+1}=\frac{a-c+c-b}{a^2+1}+\frac{b-c}{b^2+1}+\frac{c-a}{c^2+1}$$

$$=(c-a)\left(\frac{1}{c^2+1}-\frac{1}{a^2+1}\right)+(b-c)\left(\frac{1}{b^2+1}-\frac{1}{a^2+1}\right)$$

$$=-\frac{(c-a)^2(c+a)}{(c^2+1)(a^2+1)}+\frac{(b-c)(a-b)(a+b)}{(a^2+1)(b^2+1)}\le 0$$

$\boxed{\text{Case 2}}$ $a \ge b \ge c$ or $c \ge b \ge a$ and then: $\frac{a-b}{a^2+1}+\frac{b-c}{b^2+1}+\frac{c-a}{c^2+1}$

$$=\frac{a-b}{a^2+1}+\frac{b-a+a-c}{b^2+1}+\frac{c-a}{c^2+1}$$

$$=(a-b)\left(\frac{1}{a^2+1}-\frac{1}{b^2+1}\right)+(c-a)\left(\frac{1}{c^2+1}-\frac{1}{b^2+1}\right)$$

$$=-\frac{(a-b)^2(a+b)}{(a^2+1)(b^2+1)}+\frac{(c-a)(b-c)(b+c)}{(a^2+1)(b^2+1)}\le 0$$

Since we have considered all possible cases concerning the ordering of $a, b, c$ amongst each other, hence we conclude: $\frac{a-b}{a^2+1}+\frac{b-c}{b^2+1}+\frac{c-a}{c^2+1}\le 0$ $\forall\, a, b, c > 0$, " $=$ " iff $a = b = c$ (QED)