ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 1$ and $ab = 4$ then prove that :

$$\left(\frac{a-1}{b}\right)^b\left(\frac{b-1}{a}\right)^a \le \frac{1}{16}$$

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$$\text{Let } f(a) = a^a.\left(\frac{4}{a}\right)^{\frac{4}{a}}\ \forall\, a > 1 \text{ and then :}$$

$$f'(a) = a^{a-\frac{4}{a}-2}.4^{\frac{4}{a}}.\left((a^2+4)\ln a - 8\ln 2 + a^2 - 4\right)$$

$$\text{Now, since } a^{a-\frac{4}{a}-2}.4^{\frac{4}{a}} > 0\ \forall\, a > 1 \text{ and } (a^2+4)\ln a - 8\ln 2 + a^2 - 4 \lesseqgtr 0$$

$$\text{according as } a \lesseqgtr 2 \therefore f'(a) \lesseqgtr 0 \text{ according as } a \lesseqgtr 2 \Rightarrow f(a) \text{ is } \downarrow \text{ on } (1,2]$$

$$\text{and } f(a) \text{ is } \uparrow \text{ on } [2,\infty) \Rightarrow f(a) \ge f(2) = 16 \overset{ab=4}{\Rightarrow} a^a.b^b \ge 16$$

$$\Rightarrow \left(\frac{a-1}{b}\right)^b\left(\frac{b-1}{a}\right)^a \le \frac{(a-1)^b(b-1)^a}{16} \overset{?}{\le} \frac{1}{16} \Leftrightarrow b\ln(a-1) + a\ln(b-1) \underset{(*)}{\overset{?}{\le}} 0$$

$$\text{Since } \ln x \le x - 1\ \forall\, x > 0 \therefore \text{LHS of } (*) \le b(a-2) + a(b-2) \overset{ab=4}{=} 8 - 2(a+b)$$

$$\overset{AM-GM}{\le} 8 - 4.\sqrt{ab} \overset{ab=4}{=} 0 \Rightarrow (*) \text{ is true} \therefore \left(\frac{a-1}{b}\right)^b\left(\frac{b-1}{a}\right)^a \le \frac{1}{16}$$

$$\forall\, a, b > 1 \,\middle|\, ab = 4, \text{"} = \text{" iff } a = b = 2 \text{ (QED)}$$