ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0, a^2 + b^2 = 2$ then:

$$\frac{a^2}{b^3+1} + \frac{b^2}{a^3+1} \geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a^2 + b^2 = 2 \text{ or}, 2 = \frac{a^2}{1} + \frac{b^2}{1} \overset{Radon}{\geq} \frac{(a+b)^2}{2} \text{ or } a + b \leq 2\ (1)$$

$$2 = a^2 + b^2 \overset{AM-GM}{\geq} 2ab \text{ or } ab \leq 1\ (2)$$

$$\frac{a^2}{b^3+1} + \frac{b^2}{a^3+1} = \left(a^2 - \frac{a^2b^3}{b^3+1}\right) + \left(b^2 - \frac{b^2a^3}{a^3+1}\right) =$$

$$= (a^2+b^2) - a^2b^2\left(\frac{b}{b^3+1} + \frac{a}{a^3+1}\right) \overset{AM-GM}{\geq} (a^2+b^2) - a^2b^2\left(\frac{b}{2b^{\frac{3}{2}}} + \frac{a}{2a^{\frac{3}{2}}}\right) =$$

$$= (a^2+b^2) - \frac{a^2b^2}{2}\left(\sqrt{\frac{1}{b}} + \sqrt{\frac{1}{a}}\right) \overset{\substack{CBS \\ a^2+b^2=2}}{\geq} 2 - \frac{a^2b^2}{2}\sqrt{2\left(\frac{1}{b} + \frac{1}{a}\right)} =$$

$$= 2 - \frac{ab\sqrt{ab}}{2}\sqrt{2(a+b)} \overset{(1)\&(2)}{\geq} 2 - \frac{1}{2}\sqrt{4} = 1$$

Equality holds for a=b=1.