ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ *then prove that*:

$$\frac{a^3-b^3}{b^2+c^2}+\frac{b^3-c^3}{c^2+a^2}+\frac{c^3-a^3}{a^2+b^2}\geq 0$$

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$$\frac{a^3-b^3}{b^2+c^2}+\frac{b^3-c^3}{c^2+a^2}+\frac{c^3-a^3}{a^2+b^2}=$$

$$=\frac{1}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\cdot\sum_{cyc}\left((b^3-c^3)\left(b^4+\sum_{cyc}a^2b^2\right)\right)=$$

$$=\frac{1}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\cdot\left(\left(\sum_{cyc}a^2b^2\right)\sum_{cyc}(b^3-c^3)+\sum_{cyc}b^7-\sum_{cyc}b^4c^3\right)=$$

$$=\frac{1}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\cdot\left(\sum_{cyc}a^7-\sum_{cyc}b^4c^3\right)=\text{LHS}\to(*)$$

$$\text{Now, } a^7+a^7+a^7+a^7+b^7+b^7+b^7\overset{\text{AM-GM}}{\geq}7.\sqrt[7]{a^{28}b^{21}}\Rightarrow 4a^7+3b^7\overset{①}{\geq}7a^4b^3$$

$$\text{Also, } b^7+b^7+b^7+b^7+c^7+c^7+c^7\overset{\text{AM-GM}}{\geq}7.\sqrt[7]{b^{28}c^{21}}\Rightarrow 4b^7+3c^7\overset{②}{\geq}7b^4c^3$$

$$\text{Again, } c^7+c^7+c^7+c^7+a^7+a^7+a^7\overset{\text{AM-GM}}{\geq}7.\sqrt[7]{c^{28}a^{21}}\Rightarrow 4c^7+3a^7\overset{③}{\geq}7c^4a^3$$

$$\therefore ①+②+③\Rightarrow 7(a^7+b^7+c^7)\geq 7(a^4b^3+b^4c^3+c^4a^3)$$

$$\Rightarrow\sum_{cyc}a^7-\sum_{cyc}b^4c^3\overset{(**)}{\geq}0 \text{ and so, } (*) \text{ and } (**)\Rightarrow \text{LHS}\geq 0$$

$$\therefore\frac{a^3-b^3}{b^2+c^2}+\frac{b^3-c^3}{c^2+a^2}+\frac{c^3-a^3}{a^2+b^2}\geq 0\ \forall\, a,b,c>0, ''='' \ a=b=c\ (\text{QED})$$