ROMANIAN MATHEMATICAL MAGAZINE

$$If\ x,y,z>0\ and\ \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=1\ \ then\ prove$$

$$\frac{x^2y^2}{z(x^2+y^2)}+\frac{y^2z^2}{x(y^2+z^2)}+\frac{x^2z^2}{y(x^2+z^2)}\ge\frac{3\sqrt{3}}{2}$$

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Solution by Amin Hajiyev-Azerbaijan

$$Substitution\ \frac{1}{x}=a,\frac{1}{y}=b,\frac{1}{z}=c\to\ a^2+b^2+c^2=1\ \ a,b,c>0$$

$$LHS=\sum_{cyc}\frac{x^2y^2}{z(x^2+y^2)}=\sum_{cyc}\frac{1}{a^2b^2}\cdot\frac{ca^2b^2}{a^2+b^2}=\sum_{cyc}\frac{c}{a^2+b^2}$$

$$\sum_{cyc}(a^2+b^2)=\sum_{cyc}(1-c^2)$$

$$LHS=\frac{c}{1-c^2}+\frac{a}{1-a^2}+\frac{b}{1-b^2}=\sum_{cyc}\frac{c}{1-c^2}$$

$$a^2=t,b^2=u,c^2=v\to\ t+u+v=1$$

$$Convex\ function: f(k)=\frac{\sqrt{k}}{1-k},\qquad \frac{d^2f(k)}{dk^2}>0\ as\ k\in(0;1)$$

$$\frac{f(t)+f(u)+f(v)}{3}\overset{JENSEN}{\geq} f\left(\frac{u+v+t}{3}\right)$$

$$\frac{f(a^2)+f(b^2)+f(c^2)}{3}\ge f\left(\frac{a^2+b^2+c^2}{3}\right)$$

$$LHS\ge 3f\left(\frac{1}{3}\right)=3\cdot\frac{\sqrt{\frac{1}{3}}}{1-\frac{1}{3}}=\frac{9}{2\sqrt{3}}=\frac{3\sqrt{3}}{2}$$

$$\frac{x^2y^2}{z(x^2+y^2)}+\frac{y^2z^2}{x(y^2+z^2)}+\frac{x^2z^2}{y(x^2+z^2)}\ge\frac{3\sqrt{3}}{2}\quad Q.E.D$$

$$Equality\ holds\ for\ \ \frac{1}{x}=\frac{1}{y}=\frac{1}{z}=\frac{1}{\sqrt{3}}$$