ROMANIAN MATHEMATICAL MAGAZINE

$$If\ a,b,c>0\ \text{and}\ a^2+b^2+c^2\le 3\ then\ prove$$

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\ge\frac{a^2+b^2+c^2}{2}$$

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$$Cauchy-Schwarz\ inequality: \sum_{i=1}^{n}\frac{x_i^2}{y_i}\ge\frac{(\sum_{i=1}^{n}x_i)^2}{\sum_{i=1}^{n}y_i}$$

$$\begin{cases} x_1=a, x_2=b, x_3=c \\ y_1=a(b+c), y_2=b(a+c), y_3=c(a+b)\end{cases}$$

$$\frac{a^2}{a(b+c)}+\frac{b^2}{b(a+c)}+\frac{c^2}{c(a+b)}\ge\frac{(a+b+c)^2}{2(ab+ac+bc)}$$

$$LHS\ge\frac{a^2+b^2+c^2+2(ab+ac+bc)}{2(ab+ac+bc)}$$

$$LHS\ge\frac{a^2+b^2+c^2}{2(ab+ac+bc)}+1$$

$$(a-b)^2+(b-c)^2+(a-c)^2\ge 0$$

$$a^2+b^2+c^2\ge ab+ac+bc$$

$$LHS\ge\frac{1}{2}+1=\frac{3}{2}$$

$$a^2+b^2+c^2\le 3\to\ RHS=\frac{a^2+b^2+c^2}{2}\le\frac{3}{2}$$

$$LHS\ge\frac{3}{2}\ge RHS\to\ LHS\ge RHS$$

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\ge\frac{a^2+b^2+c^2}{2}\quad Q.E.D$$

$$Equality\ holds\ for\ a=b=c=1$$