ROMANIAN MATHEMATICAL MAGAZINE

$$\text{If } a,b,c>0, abc=1 \text{ then prove that}$$

$$\frac{a-1}{b^2+c^2}+\frac{b-1}{a^2+c^2}+\frac{c-1}{a^2+b^2}\geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Amin Hajiyev-Azerbaijan

$$\text{Lemma 1: Cebyshev inequality: } a_1\geq a_2\geq a_3, b_1\geq b_2\geq b_3$$

$$\frac{a_1b_1+a_2b_2+a_3b_3}{3}\geq\left(\frac{a_1+a_2+a_3}{3}\right)\left(\frac{b_1+b_2+b_3}{3}\right)$$

$$\text{Lemma 2: } a,b,c>0\rightarrow\frac{a+b+c}{3}\geq\sqrt[3]{abc}=1, a+b+c\geq 3$$

$$a\geq b\geq c\rightarrow\ a-1\geq b-1\geq c-1\ \ \frac{1}{b^2+c^2}\geq\frac{1}{a^2+c^2}\geq\frac{1}{a^2+b^2}$$

$$\frac{a-1}{b^2+c^2}+\frac{b-1}{a^2+c^2}+\frac{c-1}{a^2+b^2}\overset{Cebyshev}{\geq}\frac{1}{3}(a-1+b-1+c-1)\sum_{cyc}\frac{1}{a^2+b^2}$$

$$\sum_{cyc}\frac{a-1}{b^2+c^2}\geq\frac{1}{3}(a+b+c-3)\sum_{cyc}\frac{1}{a^2+b^2}$$

$$a+b+c-3\geq 0\rightarrow\sum_{cyc}\frac{a-1}{b^2+c^2}\geq 0\ \ Q.E.D$$

$$\text{Equality holds for }\ a=b=c=1$$