ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$ then prove that :

$$a^2+b^2+c^2+\frac{3}{a^3+b^3+c^3}\ge 4$$

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$$\sum_{cyc} a^2+\frac{3}{\sum_{cyc} a^3}\overset{abc=1}{=}\frac{\sum_{cyc} a^2}{3.\sqrt[3]{a^2b^2c^2}}+\frac{\sum_{cyc} a^2}{3.\sqrt[3]{a^2b^2c^2}}+\frac{\sum_{cyc} a^2}{3.\sqrt[3]{a^2b^2c^2}}+\frac{3abc}{\sum_{cyc} a^3}$$

$$\overset{AM-GM}{\ge} 4.\sqrt[4]{\frac{3abc\left(\sum_{cyc} a^2\right)^3}{27a^2b^2c^2\left(\sum_{cyc} a^3\right)}}\overset{?}{\ge} 4 \Leftrightarrow \left(\sum_{cyc} a^2\right)^3 \underset{(*)}{\overset{?}{\ge}} 9abc\left(\sum_{cyc} a^3\right)$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say); then : $\sum_{cyc} a = s, abc = r^2s, \sum_{cyc} a^2 = s^2-8Rr-2r^2, \sum_{cyc} a^3 = s^3-12Rrs,$

$$\text{and then, } (*) \Leftrightarrow (s^2-8Rr-2r^2)^3 \overset{?}{\ge} 9r^2s(s^3-12Rrs)$$

$$\Leftrightarrow s^6-(24Rr+15r^2)s^4+r^2(192R^2+204Rr+12r^2)s^2-8r^3(4R+r) \underset{(\bullet)}{\overset{?}{\ge}} 0$$

and $\because (s^2-16Rr+5r^2)^3 \overset{Gerretsen}{\ge} 0 \therefore$ in order to prove $(*)$, it suffices to prove :

$$\text{LHS of } (*) \overset{?}{\ge} (s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (24R-30r)s^4-r(576R^2-684Rr+63r^2)s^2+$$

$$r^2(3584R^3-4224R^2r+1104Rr^2-133r^3) \underset{(**)}{\overset{?}{\ge}} 0 \text{ and}$$

$\because (24R-30r)(s^2-16Rr+5r^2)^2 \overset{Gerretsen}{\ge} 0 \therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**) \overset{?}{\ge} (24R-30r)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (192R^2-516Rr+237r^2)s^2 \underset{(***)}{\overset{?}{\ge}} r(2560R^3-7296R^2r+4296Rr^2-617r^3)$$

Case 1 $192R^2-516Rr+237r^2 \ge 0$ and then : LHS of $(***) \overset{Gerretsen}{\ge}$ $(192R^2-516Rr+237r^2)(16Rr-5r^2) \overset{?}{\ge}$ RHS of $(***)$

$$\Leftrightarrow 128t^3-480t^2+519t-142 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(128t^2-224t+71) \overset{?}{\ge} 0$$

$$\rightarrow \text{true} \because t \overset{Euler}{\ge} 2 \Rightarrow (***) \text{ is true}$$

$\boxed{Case\ 2}$ $192R^2 - 516Rr + 237r^2 < 0$ *and* then : LHS of $(***) \overset{Gerretsen}{\geq}$

$$(192R^2 - 516Rr + 237r^2)(4R^2 + 4Rr + 3r^2) \overset{?}{\geq} \text{RHS of } (***)$$

$$\Leftrightarrow 192t^4 - 964t^3 + 1689t^2 - 1224t + 332 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left((t-2)(192t^2 - 196t + 137) + 108\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{Euler}{\geq} 2$$

$\Rightarrow (***)$ is true $\therefore$ combining both cases, $(***) \Rightarrow (**) \Rightarrow (*)$ is true $\forall\, \Delta\, XYZ_{s,R,r}$

$$\Rightarrow (\bullet) \text{ is true } \therefore a^2 + b^2 + c^2 + \frac{3}{a^3 + b^3 + c^3} \geq 4\ \forall\, a, b, c > 0 \,\middle|\, abc = 1,$$

$'' ='' $ iff $a = b = c = 1$ (QED)