ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then:

$$\frac{a^6+b^6+c^6}{a^2b^2c^2}+\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2 \ge 4$$

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Solution by Tapas Das-India

$$ab+bc+ca \overset{AM-GM}{\ge} 3\sqrt[3]{a^2b^2c^2}\ or, \frac{(ab+bc+ca)^3}{27} \ge a^2b^2c^2\ (1)$$

$$a^6+b^6+c^6=\frac{(a^2)^3}{(1)^2}+\frac{(b^2)^3}{(1)^2}+\frac{(c^2)^3}{(1)^2} \overset{Radon}{\ge} \frac{(a^2+b^2+c^2)^3}{9}\ (2)$$

$$Let\ x=\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1\ as\ a^2+b^2+c^2 \ge\ ab+bc+ca$$

$$We\ need\ to\ show:$$

$$\frac{a^6+b^6+c^6}{a^2b^2c^2}+\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2 \ge 4$$

$$\frac{\frac{(a^2+b^2+c^2)^3}{9}}{\frac{(ab+bc+ca)^3}{27}}+\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2 \ge 4$$

$$3\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^3+\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2 \ge 4$$

$$3x^3+\frac{1}{x^2} \ge 4\ or, 3x^5-4x^2+1 \ge 0$$

$$(x-1)(3x^4+3x^3+3x^2-1) \ge 0\ true\ as\ x \ge 1$$

Equality holds for a=b=c