ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$ then prove that :

$$\frac{a-1}{\sqrt{b+c}}+\frac{b-1}{\sqrt{c+a}}+\frac{c-1}{\sqrt{a+b}} \geq 0$$

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$$\sum_{cyc}\frac{a-1}{\sqrt{b+c}} = \sum_{cyc}\frac{a^2}{a.\sqrt{b+c}} - \sum_{cyc}\frac{\sqrt{(c+a)(a+b)}}{\sqrt{(b+c)(c+a)(a+b)}} \overset{\substack{\text{Bergstrom} \\ and \\ \text{CBS}}}{\geq}$$

$$\frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc}\left(\sqrt{a}.\sqrt{ab+ac}\right)} - \frac{1}{\sqrt{(b+c)(c+a)(a+b)}}.\sqrt{\sum_{cyc}(c+a)}.\sqrt{\sum_{cyc}(a+b)}$$

$$\overset{\text{CBS}}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{\sqrt{\sum_{cyc} a}.\sqrt{2\sum_{cyc} ab}} - \frac{2\sum_{cyc} a}{\sqrt{(b+c)(c+a)(a+b)}} \overset{\text{AM-GM}}{\geq}$$

$$\frac{3\sum_{cyc} a}{\sqrt{2}.\sqrt{\prod_{cyc}(b+c)+abc}} - \frac{2\sum_{cyc} a}{\sqrt{\prod_{cyc}(b+c)}} \left(\because \sum_{cyc} a \overset{\text{AM-GM}}{\geq} 3.\sqrt[3]{abc} \overset{abc=1}{=} 3\right)$$

$$= \left(\sum_{cyc} a\right).\frac{9\prod_{cyc}(b+c) - 8\left(\prod_{cyc}(b+c)+abc\right)}{\left(2\left(\prod_{cyc}(b+c)+abc\right)+\prod_{cyc}(b+c)\right)\left(\frac{3}{\sqrt{2}.\sqrt{\prod_{cyc}(b+c)+abc}}+\frac{2}{\sqrt{\prod_{cyc}(b+c)}}\right)}$$

$$= \left(\sum_{cyc} a\right).\frac{\prod_{cyc}(b+c) - 8abc}{\left(2\left(\prod_{cyc}(b+c)+abc\right)+\prod_{cyc}(b+c)\right)\left(\frac{3}{\sqrt{2}.\sqrt{\prod_{cyc}(b+c)+abc}}+\frac{2}{\sqrt{\prod_{cyc}(b+c)}}\right)}$$

$$\geq 0 \text{ via Cesaro } \therefore \frac{a-1}{\sqrt{b+c}}+\frac{b-1}{\sqrt{c+a}}+\frac{c-1}{\sqrt{a+b}} \geq 0 \;\forall\, a,b,c > 0 \,\middle|\, abc = 1,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$