ROMANIAN MATHEMATICAL MAGAZINE

If $x, y > 0, x \neq y$ then:

$$\left(\frac{(x+y)xy}{(x-y)^2}+x+y\right)\left(\frac{x+y}{(x-y)^2}+\frac{1}{x}+\frac{1}{y}\right)\frac{(x+y)^2}{xy} \geq 81$$

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Solution by Tapas Das-India

$$Let\ a = x+y\ ,b = xy\ then\ (x-y)^2 = (x+y)^2 - 4xy = a^2 - 4b$$

$$a^2 = (x+y)^2 \overset{AM-GM}{>} 4xy\ or\ a^2 > 4b\ or\ \frac{a^2}{b} > 4\ or\ t \overset{t=\frac{a^2}{b}}{=} 4\ (1)$$

$$\frac{(x+y)xy}{(x-y)^2}+x+y = \frac{ab}{a^2-4b}+a\ and\ \frac{x+y}{(x-y)^2}+\frac{1}{x}+\frac{1}{y} = \frac{a}{a^2-4b}+\frac{a}{b}, \frac{(x+y)^2}{xy} = \frac{a^2}{b}$$

$$L.H.S = \left(\frac{ab}{a^2-4b}+a\right)\left(\frac{a}{a^2-4b}+\frac{a}{b}\right).\frac{a^2}{b} = \frac{a^4}{b^2}.\frac{(a^2-3b)^2}{(a^2-4b)^2} \overset{t=\frac{a^2}{b}}{=} \frac{t^2(t-3)^2}{(t-4)^2}$$

$$We\ need\ to\ show:$$

$$\frac{t^2(t-3)^2}{(t-4)^2} \geq 81\ or, \frac{t(t-3)}{t-4} \geq 9\ or, \frac{t(t-3)}{t-4} - 9 \geq 0\ or, \frac{(t-6)^2}{t-4} \geq 0$$

$$which\ is\ true\ as\ t > 4\big(from\ (1)\big)$$

$$and\ equality\ occurs\ when\ t = 6\ or, \frac{a^2}{b} = 6\ or, a^2 = 6b$$