ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0$ and $x+y+z=1$ then prove that :

$$\sum_{cyc}\frac{az+by}{(a+b)(x+yz)}\ge\sum_{cyc}\frac{x}{x+yz}\text{ for all } a+b>0$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{az+by}{(a+b)(x+yz)}-\sum_{cyc}\frac{x}{x+yz}\overset{x+y+z=1}{=}$$

$$\sum_{cyc}\frac{az+by}{(a+b)(1-y-z+yz)}-\sum_{cyc}\frac{x}{1-y-z+yz}$$

$$=\sum_{cyc}\frac{az+by}{(a+b)(1-y)(1-z)}-\sum_{cyc}\frac{x}{(1-y)(1-z)}$$

$$\overset{x+y+z=1}{=}\sum_{cyc}\frac{az+by}{(a+b)(z+x)(x+y)}-\sum_{cyc}\frac{x}{(z+x)(x+y)}$$

$$=\frac{1}{(x+y)(y+z)(z+x)}\cdot\left(\sum_{cyc}\frac{(az+by)(y+z)}{a+b}-\sum_{cyc}(x(y+z))\right)$$

$$=\frac{1}{(x+y)(y+z)(z+x)}\cdot\left(\frac{a+b}{a+b}\cdot\left(\sum_{cyc}x^2+\sum_{cyc}xy\right)-2\sum_{cyc}xy\right)$$

$$=\frac{1}{(x+y)(y+z)(z+x)}\cdot\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)\ (\because a+b\ne 0 \text{ as } a+b>0)\ge 0$$

$$(\because x,y,z>0\Rightarrow(x+y)(y+z)(z+x)>0)\ \therefore\sum_{cyc}\frac{az+by}{(a+b)(x+yz)}\ge\sum_{cyc}\frac{x}{x+yz}$$

$$\forall\, x,y,z>0\,\big|\,x+y+z=1 \text{ and } \forall\, a,b\,\big|\,a+b\ne 0,\text{'' = ''}\ \text{iff } x=y=z=\frac{1}{3}\ (\text{QED})$$