ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$ then prove the inequality :

$$\sum_{cyc} \frac{8}{(a+c)b + a + b + 4} \le 3$$

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$$\sum_{cyc} \frac{8}{(a+c)b + a + b + 4} \overset{abc=1}{=} \sum_{cyc} \frac{8}{ab + \left(\frac{1}{a} + a\right) + b + 4}$$

$$\overset{AM-GM}{\le} \sum_{cyc} \frac{8}{ab + b + 2 + 4} \overset{AM-GM}{\le} \sum_{cyc} \frac{8}{2\sqrt{ab^2} + 6} \overset{abc=1}{=} \sum_{cyc} \frac{4}{\sqrt{\frac{b}{c}} + 3} = \sum_{cyc} \frac{4}{x+3}$$

$$\left(x = \sqrt{\frac{b}{c}}, y = \sqrt{\frac{c}{a}}, z = \sqrt{\frac{a}{b}}\right) = \sum_{cyc} \frac{4(y+3)(z+3)}{(x+3)(y+3)(z+3)} \overset{?}{\le} 3$$

$$\Leftrightarrow 3xyz + 5\sum_{cyc} xy + 3\sum_{cyc} x \overset{?}{\ge} 27 \Leftrightarrow 5\sum_{cyc} xy + 3\sum_{cyc} x \underset{(*)}{\overset{?}{\ge}} 24 \ (\because xyz = 1)$$

$$\text{Indeed, } 5\sum_{cyc} xy + 3\sum_{cyc} x \overset{AM-GM}{\ge} 15.\sqrt[3]{x^2y^2z^2} + 9.\sqrt[3]{xyz} \overset{xyz=1}{=} 24 \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{8}{(a+c)b + a + b + 4} \le 3 \ \forall \ a, b, c > 0 \mid abc = 1,$$

$$'' ='' \text{ iff } a = b = c = 1$$