ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, \sum_{cyc} \frac{\lambda+1}{\lambda^2+x^2} = 3$ and $\lambda \geq 1$ then :

$$\sum_{cyc} \frac{\lambda - x}{\lambda^2 - \lambda x + x^2} \geq 0$$

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$$\frac{1}{\lambda}.\sum_{cyc} \frac{(\lambda^2 - \lambda x + x^2) - x^2}{\lambda^2 - \lambda x + x^2} = \frac{1}{\lambda}.\left(3 - \sum_{cyc} \frac{x^2}{\lambda^2 - \lambda x + x^2}\right)$$

$$\overset{AM-GM}{\geq} \frac{1}{\lambda}.\left(3 - \sum_{cyc} \frac{x^2}{\lambda^2 - \frac{\lambda^2+x^2}{2} + x^2}\right) = \frac{1}{\lambda}.\left(3 - \sum_{cyc} \frac{2(x^2+\lambda^2-\lambda^2)}{\lambda^2+x^2}\right)$$

$$= \frac{1}{\lambda}.\left(-3 + 2\lambda^2.\sum_{cyc} \frac{1}{\lambda^2+x^2}\right) = \frac{1}{\lambda}.\left(-3 + 2\lambda^2.\frac{3}{\lambda+1}\right) \left(\because \sum_{cyc} \frac{\lambda+1}{\lambda^2+x^2} = 3\right)$$

$$= \frac{3}{\lambda(\lambda+1)}.(2\lambda^2 - \lambda - 1) = \frac{3(\lambda-1)(2\lambda+1)}{\lambda(\lambda+1)} \geq 0 \ (\because \lambda \geq 1)$$

$$\therefore \sum_{cyc} \frac{\lambda - x}{\lambda^2 - \lambda x + x^2} \geq 0 \ \forall \ x, y, z > 0 \left| \sum_{cyc} \frac{\lambda+1}{\lambda^2+x^2} = 3 \text{ and } \lambda \geq 1, \right.$$

$$'' ='' \text{ iff } x = y = z = \lambda = 1 \text{ (QED)}$$