ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, \sum_{cyc} x = 1$ and $\lambda \geq \frac{2}{3}$ then :

$$\lambda \sum_{cyc} x^3 + xyz \geq \frac{3\lambda + 1}{9} \sum_{cyc} x^2$$

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$$\lambda \sum_{cyc} x^3 + xyz - \frac{3\lambda + 1}{9} \sum_{cyc} x^2 =$$

$$\frac{\lambda}{3}\left(3\sum_{cyc} x^3 - \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right)\right) + \frac{1}{9}\left(9xyz - \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right)\right) \left(\because \sum_{cyc} x = 1\right)$$

$$\overset{\lambda \geq \frac{2}{3}}{\geq} \frac{2}{9}\left(3\sum_{cyc} x^3 - \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right)\right) + \frac{1}{9}\left(9xyz - \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right)\right)$$

$$\left(\because \sum_{cyc} x^3 \overset{\text{Chebyshev}}{\geq} \frac{1}{3}\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right)\right)$$

$$= \frac{1}{3}\left(2\sum_{cyc} x^3 - \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right) + 3xyz\right)$$

$$= \frac{1}{3}\left(\sum_{cyc} x^3 + 3xyz - \sum_{cyc} x^2 y - \sum_{cyc} xy^2\right) \overset{\text{Schur}}{\geq} 0$$

$$\therefore \lambda \sum_{cyc} x^3 + xyz \geq \frac{3\lambda + 1}{9} \sum_{cyc} x^2 \ \forall\, x, y, z > 0 \,\Big|\, \sum_{cyc} x = 1 \text{ and } \lambda \geq \frac{2}{3},$$

$$'' ='' \text{ iff } x = y = z = \frac{1}{3} \text{ (QED)}$$