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If $a,b,c>0, abc=1$ then :

$$\sum_{cyc}\sqrt[4]{\frac{2a}{a+1}}\le 3$$

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Since $abc=1$, we can assign : $a=\frac{yz}{x^2}, b=\frac{zx}{y^2}, c=\frac{xy}{z^2}$ and then :

$$\sum_{cyc}\sqrt[4]{\frac{2a}{a+1}}\overset{CBS}{\le}\sqrt{3}.\sqrt{\sum_{cyc}\sqrt{\frac{2yz}{yz+x^2}}}\overset{?}{\le}3$$

$$\Leftrightarrow \sum_{cyc}\sqrt{yz(zx+y^2)(xy+z^2)}\underset{(*)}{\overset{?}{\le}}\frac{3}{\sqrt{2}}.\sqrt{(yz+x^2)(zx+y^2)(xy+z^2)}$$

$$\text{Now, }\sum_{cyc}\sqrt{yz(zx+y^2)(xy+z^2)}\overset{CBS}{\le}\sqrt{\sum_{cyc}y(xy+z^2)}.\sqrt{\sum_{cyc}z(zx+y^2)}$$

$$=2\sqrt{\left(\sum_{cyc}xy^2\right)\left(\sum_{cyc}x^2y\right)}\overset{?}{\le}\frac{3}{\sqrt{2}}.\sqrt{(yz+x^2)(zx+y^2)(xy+z^2)}$$

$$\Leftrightarrow 9\left(xyz\sum_{cyc}x^3+\sum_{cyc}x^3y^3+2x^2y^2z^2\right)\overset{?}{\ge}8\left(xyz\sum_{cyc}x^3+\sum_{cyc}x^3y^3+3x^2y^2z^2\right)$$

$$\Leftrightarrow xyz\sum_{cyc}x^3+\sum_{cyc}x^3y^3\overset{?}{\ge}6x^2y^2z^2\to\text{true}\because xyz\sum_{cyc}x^3+\sum_{cyc}x^3y^3\overset{AM-GM}{\ge}$$

$$3x^2y^2z^2+3x^2y^2z^2=6x^2y^2z^2\Rightarrow(*)\text{ is true}\therefore\sum_{cyc}\sqrt[4]{\frac{2a}{a+1}}\le 3\ \forall\ abc=1,$$

$$''=''\ a=b=c=1\ (QED)$$