ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x^3 + y^3 + z^3 = 3$ then :

$$\sum_{cyc}\left(\frac{x^2+1}{x+1}\right)^3 \geq 3$$

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$$\left(\frac{x^2+1}{x+1}\right)^3 \overset{?}{\geq} \frac{x^3+1}{2} \Leftrightarrow 2(x^2+1)^3 \overset{?}{\geq} (x^3+1)(x+1)^3$$

$$\Leftrightarrow (x^2+x+1)(x-1)^4 \overset{?}{\geq} 0 \to \text{true} \therefore \left(\frac{x^2+1}{x+1}\right)^3 \geq \frac{x^3+1}{2} \textit{ and analogs}$$

$$\Rightarrow \sum_{cyc}\left(\frac{x^2+1}{x+1}\right)^3 \geq \sum_{cyc}\frac{x^3+1}{2} = \frac{1}{2}(3) + \frac{3}{2}\left(\because \sum_{cyc} x^3 = 3\right) = 3$$

$$\text{So, } \sum_{cyc}\left(\frac{x^2+1}{x+1}\right)^3 \geq 3 \ \forall\, x, y, z > 0 \,\middle|\, x^3+y^3+z^3 = 3, '' ='' \ x = y = z = 1 \text{ (QED)}$$