ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, xyz \geq 1$ *and* $n \in \mathbb{N}^*$ then :

$$\sum_{cyc} \frac{x^{9n-1}}{x^2+1} \geq \frac{3}{2}(xyz)^{3n-1}$$

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$$\sum_{cyc} \frac{x^{9n-1}}{x^2+1} = \sum_{cyc} \frac{x^{9n-3}.(x^2+1-1)}{x^2+1} = \sum_{cyc} x^{9n-3} - \sum_{cyc} \frac{x^{9n-3}}{x^2+1}$$

$$\overset{AM-GM}{\geq} \frac{1}{2}\sum_{cyc} x^{9n-3} + \frac{1}{2}\sum_{cyc} x^{9n-3} - \sum_{cyc} \frac{x^{9n-3}}{2x} \overset{AM-GM}{\geq}$$

$$\frac{3}{2}.\sqrt[3]{\prod_{cyc} x^{9n-3}} + \frac{1}{2}\sum_{cyc} x^{9n-4}.x - \frac{1}{2}\sum_{cyc} x^{9n-4}$$

$$\overset{Chebyshev}{\geq} \frac{3}{2}(xyz)^{3n-1} + \frac{1}{6}\left(\sum_{cyc} x^{9n-4}\right)\left(\sum_{cyc} x\right) - \frac{1}{2}\sum_{cyc} x^{9n-4}$$

$$\left(\because \text{WLOG } \textit{assuming } x \geq y \geq z \Rightarrow x^{9n-4} \geq y^{9n-4} \geq z^{9n-4} \textit{ as } n \in \mathbb{N}^* \Rightarrow (9n-4) > 0\right)$$

$$\overset{AM-GM}{\geq} \frac{3}{2}(xyz)^{3n-1} + \frac{1}{6}\left(\sum_{cyc} x^{9n-4}\right)(3.\sqrt[3]{xyz}) - \frac{1}{2}\sum_{cyc} x^{9n-4}$$

$$\overset{xyz \geq 1}{\geq} \frac{3}{2}(xyz)^{3n-1} + \frac{1}{2}\sum_{cyc} x^{9n-4} - \frac{1}{2}\sum_{cyc} x^{9n-4} = \frac{3}{2}(xyz)^{3n-1} \text{ and so,}$$

$$\sum_{cyc} \frac{x^{9n-1}}{x^2+1} \geq \frac{3}{2}(xyz)^{3n-1} \;\forall\, x, y, z > 0 \mid xyz \geq 1 \textit{ and } n \in \mathbb{N}^*,$$

$$'' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$