ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, a+b+c=1$ and $0 \le \lambda \le 4$ then :

$$\lambda \sum_{cyc} ab \le 1 + 9(\lambda - 3)abc$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\lambda \sum_{cyc} ab \overset{?}{\le} 1 + 9(\lambda - 3)abc$$

$$\Leftrightarrow \lambda\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc\right) \overset{?}{\le} \left(\sum_{cyc} a\right)^3 - 27abc \text{ and } \because 0 \le \lambda \le 4$$

$$\text{and } \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) \overset{\text{AM-GM}}{\ge} 9abc \therefore \lambda\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc\right) \le$$

$$4\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc\right) \overset{?}{\le} \left(\sum_{cyc} a\right)^3 - 27abc$$

$$\Leftrightarrow \sum_{cyc} a^3 + 3abc \overset{?}{\ge} \sum_{cyc} a^2b + \sum_{cyc} ab^2 \to \text{true via Schur}$$

$$\therefore \lambda \sum_{cyc} ab \le 1 + 9(\lambda - 3)abc \ \forall\, a, b, c > 0 \,\middle|\, a+b+c=1 \text{ and } 0 \le \lambda \le 4,$$

$$'' = '' \text{ iff } a = b = c = \frac{1}{3} \text{ (QED)}$$