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If $a, b, c > 0, a + b + c = 1$ and $\lambda \geq 2$ then :

$$\sum_{cyc} a^2 \leq (\lambda + 1)\sqrt{\frac{\sum_{cyc} a^2}{3}} - \lambda\sqrt{3abc}$$

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$$1 = \sum_{cyc} a \overset{AM-GM}{\geq} 3.\sqrt[3]{abc} \Rightarrow abc \leq \frac{1}{27} \Rightarrow \sqrt{3abc} \leq \frac{1}{3} \rightarrow (1) \text{ and also,}$$

$$\sqrt{\frac{\sum_{cyc} a^2}{3}} \geq \sqrt{\frac{\left(\sum_{cyc} a\right)^2}{9}} \Rightarrow \sqrt{\frac{\sum_{cyc} a^2}{3}} \geq \frac{1}{3} \rightarrow (2) \left(\because \sum_{cyc} a = 1\right) \therefore (1) \text{ and } (2) \Rightarrow$$

$$\sqrt{\frac{\sum_{cyc} a^2}{3}} - \sqrt{3abc} \geq 0 \rightarrow (3) \text{ and now, } (\lambda + 1).\sqrt{\frac{\sum_{cyc} a^2}{3}} - \lambda.\sqrt{3abc}$$

$$= \lambda\left(\sqrt{\frac{\sum_{cyc} a^2}{3}} - \sqrt{3abc}\right) + \sqrt{\frac{\sum_{cyc} a^2}{3}} \overset{\substack{\lambda \geq 2 \\ \text{and} \\ \text{via } (3)}}{\geq} 2\left(\sqrt{\frac{\sum_{cyc} a^2}{3}} - \sqrt{3abc}\right) + \sqrt{\frac{\sum_{cyc} a^2}{3}}$$

$$\overset{a+b+c=1}{=} \sqrt{3\sum_{cyc} a^2} - 2.\sqrt{3abc\left(\sum_{cyc} a\right)} \geq \sqrt{3\sum_{cyc} a^2} - 2\sum_{cyc} ab$$

$$= \sqrt{3\sum_{cyc} a^2} - \left(2\sum_{cyc} ab + \sum_{cyc} a^2\right) + \sum_{cyc} a^2 \geq \sum_{cyc} a - \left(\sum_{cyc} a\right)^2 + \sum_{cyc} a^2$$

$$\overset{a+b+c=1}{=} \sum_{cyc} a^2 \text{ and so, } \sum_{cyc} a^2 \leq (\lambda + 1).\sqrt{\frac{\sum_{cyc} a^2}{3}} - \lambda.\sqrt{3abc}$$

$$\forall a, b, c > 0 \,\big|\, a + b + c = 1 \text{ and } \lambda \geq 2, '' ='' \text{ iff } a = b = c = \frac{1}{3} \text{ (QED)}$$