ROMANIAN MATHEMATICAL MAGAZINE

$$\text{If } x,y,z>0,\ xy+yz+zx=2 \text{ and } \lambda\ge\frac{8}{9} \text{ then :}$$

$$\lambda xyz(x+y+z)-x^2y^2z^2\le\frac{4}{27}(9\lambda-2)$$

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$$xyz\left(\sum_{cyc}x\right)\le\frac{1}{3}\left(\sum_{cyc}xy\right)^2=\frac{4}{3}\Rightarrow\frac{4}{3}-xyz\left(\sum_{cyc}x\right)\ge 0\to (1)\text{ and now,}$$

$$xyz\left(\sum_{cyc}x\right)-x^2y^2z^2\overset{?}{\le}\frac{4}{27}(9\lambda-2)\Leftrightarrow\lambda\left(\frac{4}{3}-xyz\left(\sum_{cyc}x\right)\right)-\frac{8}{27}+x^2y^2z^2\overset{?}{\ge}0$$

$$\text{and }\because\frac{4}{3}-xyz\left(\sum_{cyc}x\right)\overset{\text{via }(1)}{\ge}0\text{ and }\because\lambda\ge\frac{8}{9}\therefore\text{ it suffices to prove :}$$

$$\frac{8}{9}\left(\frac{4}{3}-xyz\left(\sum_{cyc}x\right)\right)-\frac{8}{27}+x^2y^2z^2\overset{?}{\ge}0$$

$$\Leftrightarrow\frac{4}{9}\left(\sum_{cyc}xy\right)\left(\frac{1}{3}\left(\sum_{cyc}xy\right)^2-xyz\left(\sum_{cyc}x\right)\right)-\frac{1}{27}\left(\sum_{cyc}xy\right)^3+x^2y^2z^2\overset{?}{\ge}0$$

$$\left(\because\sum_{cyc}xy=2\right)\Leftrightarrow\left(\sum_{cyc}xy\right)^3+9x^2y^2z^2\overset{?}{\ge}4xyz\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)$$

$$\Leftrightarrow\sum_{cyc}x^3y^3+3x^2y^2z^2\overset{?}{\ge}xyz\left(\sum_{cyc}x^2y+\sum_{cyc}xy^2\right)\to\text{true}$$

$$\because\sum_{cyc}a^3+3abc\overset{\text{Schur}}{\ge}\sum_{cyc}a^2b+\sum_{cyc}ab^2\text{ and choosing }a\equiv xy,b\equiv yz,c\equiv zx$$

$$\therefore\lambda xyz(x+y+z)-x^2y^2z^2\le\frac{4}{27}(9\lambda-2)\ \forall\ x,y,z>0\ \big|\ xy+yz+zx=2\text{ and}$$

$$\lambda\ge\frac{8}{9},''=''\text{ iff }x=y=z=\sqrt{\frac{2}{3}}\ \text{(QED)}$$