ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0,\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$ *and* $27\lambda+8n=36,\lambda\ge 1,n>0$ *then* :

$$\sum_{cyc} x \le \lambda xyz + n$$

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$$2=\sum_{cyc}\frac{1}{x}\overset{AM-GM}{\ge}\frac{3}{\sqrt[3]{xyz}}\Rightarrow 8xyz\ge 27\rightarrow (1) \text{ and now, } \sum_{cyc}x\overset{?}{\le}\lambda xyz+n$$

$$\Leftrightarrow 8\sum_{cyc}x\overset{?}{\le}8\lambda xyz+36-27\lambda\ (\because 27\lambda+8n=36)$$

$$\Leftrightarrow 8\sum_{cyc}x\overset{?}{\le}\lambda(8xyz-27)+36 \text{ and } \because 8xyz-27\overset{\text{via }(1)}{\ge}0 \text{ and } \because \lambda\ge 1$$

$$\therefore \text{ it suffices to prove : } 8\sum_{cyc}x\overset{?}{\le}8xyz-27+36$$

$$\Leftrightarrow 8\sum_{cyc}x\overset{?}{\le}8xyz\left(\frac{\sum_{cyc}xy}{2xyz}\right)^2+9\left(\frac{2xyz}{\sum_{cyc}xy}\right)\ \left(\because \frac{\sum_{cyc}xy}{2xyz}=1\right)$$

$$\Leftrightarrow 4xyz\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\overset{?}{\le}\left(\sum_{cyc}xy\right)^3+9x^2y^2z^2$$

$$\Leftrightarrow \sum_{cyc}x^3y^3+3x^2y^2z^2\overset{?}{\ge}xyz\left(\sum_{cyc}x^2y+\sum_{cyc}xy^2\right)\rightarrow \text{true}$$

$$\because \sum_{cyc}a^3+3abc\overset{Schur}{\ge}\sum_{cyc}a^2b+\sum_{cyc}ab^2 \text{ and choosing } a\equiv xy, b\equiv yz, c\equiv zx$$

$$\therefore \sum_{cyc}x\le\lambda xyz+n\ \forall\ x,y,z>0 \mid \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2 \text{ and } 27\lambda+8n=36,\lambda\ge 1,n>0,$$

$$''='' \text{ iff } x=y=z=\frac{3}{2}\ (QED)$$