ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, xyz = 1$ and $\lambda \geq 0$ then :

$$\sum_{cyc} \sqrt{x^2 - xy + y^2} \leq \lambda \left(\sum_{cyc} x^2\right)^2 + (1 - 3\lambda)\left(\sum_{cyc} x^2\right)$$

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$$\sqrt{\left(\sum_{cyc} xy\right)\left(2\sum_{cyc} x^2 - \sum_{cyc} xy\right)} \overset{AM-GM}{\leq} \sum_{cyc} x^2$$

$$\Rightarrow \sum_{cyc} x^2 \geq \sqrt{\frac{\sum_{cyc} xy}{3}} \cdot \sqrt{\sum_{cyc}(1)\left(\sum_{cyc}(x^2 - xy + y^2)\right)} \overset{\text{Reverse CBS}}{\geq}$$

$$\sqrt{\frac{\sum_{cyc} xy}{3}} \cdot \sum_{cyc} \sqrt{x^2 - xy + y^2} \geq \sum_{cyc} \sqrt{x^2 - xy + y^2}$$

$$\left(\because \sum_{cyc} xy \overset{AM-GM}{\geq} 3.\sqrt[3]{x^2y^2z^2} \overset{xyz=1}{=} 3\right)$$

$$\therefore \sum_{cyc} x^2 + \lambda\left(\sum_{cyc} x^2\right)^2 \geq \sum_{cyc} \sqrt{x^2 - xy + y^2} + \lambda\left(\sum_{cyc} x^2\right)^2 \geq$$

$$\sum_{cyc} \sqrt{x^2 - xy + y^2} + 3\lambda\left(\sum_{cyc} x^2\right)\left(\because \sum_{cyc} x^2 \overset{AM-GM}{\geq} 3.\sqrt[3]{x^2y^2z^2} \overset{xyz=1}{=} 3 \text{ and } \because \lambda \geq 0\right)$$

$$\therefore \sum_{cyc} \sqrt{x^2 - xy + y^2} \leq \lambda\left(\sum_{cyc} x^2\right)^2 + (1 - 3\lambda)\left(\sum_{cyc} x^2\right)$$

$\forall\, x, y, z > 0 \,\middle|\, xyz = 1$ and $\lambda \geq 0$, " = " iff $x = y = z = 1$ (QED)