ROMANIAN MATHEMATICAL MAGAZINE

If $a,\mathrm{b},\mathrm{c}\in[0,1]$ and $\lambda\geq 7$ then :

$$\sum_{cyc}\frac{a}{\mathrm{b}+\mathrm{c}^3+\lambda}\leq\frac{3}{\lambda+2}$$

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$$a,\mathrm{b},\mathrm{c}\in[0,1]\Rightarrow \mathrm{b}\geq\mathrm{b}^3\text{ and analogs}\therefore\sum_{cyc}\frac{a}{\mathrm{b}+\mathrm{c}^3+\lambda}\leq\sum_{cyc}\frac{a}{\mathrm{b}^3+\mathrm{c}^3+\lambda}$$

$$=\sum_{cyc}\frac{a}{\lambda+2-(1-\mathrm{b}^3)-(1-\mathrm{c}^3)}=\sum_{cyc}\frac{\sqrt[3]{1-x}}{\mathrm{m}-\mathrm{y}-\mathrm{z}}$$

$$(x=1-a^3,\mathrm{y}=1-\mathrm{b}^3,\mathrm{z}=1-\mathrm{c}^3,\mathrm{m}=\lambda+2)\leq\sum_{cyc}\frac{\frac{3-x}{3}}{\mathrm{m}-\mathrm{y}-\mathrm{z}}$$

$$\left(\begin{array}{c}\because\sqrt[3]{1-x}\overset{?}{\leq}\frac{3-x}{3}\Leftrightarrow(3-x)^3\overset{?}{\geq}27(1-x)\Leftrightarrow x^2(9-x)\overset{?}{\geq}0\\ \to\text{true}\because 0\leq a^3,\mathrm{b}^3,\mathrm{c}^3\leq 1\Rightarrow 0\leq x,\mathrm{y},\mathrm{z}\leq 1\therefore\sqrt[3]{1-x}\leq\frac{3-x}{3}\text{ and analogs}\end{array}\right)$$

$$\overset{?}{\leq}\frac{3}{\lambda+2}=\frac{3}{\mathrm{m}}\Leftrightarrow\sum_{cyc}\frac{\mathrm{m}(3-x)}{\mathrm{m}-\mathrm{y}-\mathrm{z}}\underset{(*)}{\overset{?}{\leq}}9$$

$$\text{Now, }\sum_{cyc}\frac{\mathrm{m}(3-x)}{\mathrm{m}-\mathrm{y}-\mathrm{z}}=\sum_{cyc}\frac{(\mathrm{m}-\mathrm{y}-\mathrm{z}+\mathrm{y}+\mathrm{z})(3-x)}{\mathrm{m}-\mathrm{y}-\mathrm{z}}$$

$$=\sum_{cyc}(3-x)+\sum_{cyc}\frac{(\mathrm{y}+\mathrm{z})(3-x)}{\mathrm{m}-\mathrm{y}-\mathrm{z}}\leq 9-\sum_{cyc}x+\sum_{cyc}\frac{(\mathrm{y}+\mathrm{z})(3-x)}{7}$$

$$\left(\begin{array}{c}\because 0\leq\mathrm{y},\mathrm{z}\leq 1\Rightarrow\mathrm{m}-\mathrm{y}-\mathrm{z}\geq\mathrm{m}-2\geq 9-2=7\text{ as }\mathrm{m}=\lambda+2\geq 9\\ \text{and}\because 0\leq x,\mathrm{y},\mathrm{z}\leq 1\Rightarrow\mathrm{y}+\mathrm{z}\geq 0\text{ and }3-x>0\end{array}\right)$$

$$\leq 9-\sum_{cyc}x+\frac{1}{7}.\left(6\sum_{cyc}x-2\sum_{cyc}xy\right)=9-\frac{1}{7}.\left(\sum_{cyc}x+2\sum_{cyc}xy\right)\leq 9$$

$$(\because x,\mathrm{y},\mathrm{z}\geq 0)\Rightarrow(*)\text{ is true}\therefore\sum_{cyc}\frac{a}{\mathrm{b}+\mathrm{c}^3+\lambda}\leq\frac{3}{\lambda+2}\ \forall a,\mathrm{b},\mathrm{c}\in[0,1]\text{ and }\lambda\geq 7,$$

$$''=''\text{ iff }(a=a^3\text{ and analogs})\wedge(x=1-a^3=\mathrm{y}=1-\mathrm{b}^3=\mathrm{z}=1-\mathrm{c}^3=0)$$

$$\Rightarrow''=''\text{ iff }a=\mathrm{b}=\mathrm{c}=1\text{ (QED)}$$