ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, abc = 1$ and $\lambda \geq 1$ then :

$$\sum_{cyc} \frac{a+b}{a+b+\lambda} \geq \frac{6}{\lambda+2}$$

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$$\sum_{cyc} \frac{a+b}{a+b+\lambda} = \sum_{cyc} \frac{x+\lambda-\lambda}{x+\lambda} \ (x = b+c, y = c+a, z = a+b)$$

$$= 3 - \frac{\lambda\left(\sum_{cyc} xy + 3\lambda^2 + 2\lambda \sum_{cyc} x\right)}{xyz + \lambda^3 + \lambda \sum_{cyc} xy + \lambda^2 \sum_{cyc} x} \overset{?}{\geq} \frac{6}{\lambda+2}$$

$$\Leftrightarrow 3xyz + 2(\lambda-1)\left(\sum_{cyc} xy\right) + (\lambda^2 - 4\lambda)\left(\sum_{cyc} x\right) - 6\lambda^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow 3\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3 + 2(\lambda-1)\left(\left(\sum_{cyc} a\right)^2 + \sum_{cyc} ab\right) +$$

$$(2\lambda^2 - 8\lambda)\left(\sum_{cyc} a\right) - 6\lambda^2 \overset{?}{\underset{(*)}{\geq}} 0$$

$$\text{Now, LHS of } (*) \overset{AM-GM}{\geq} 2\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) + 9abc - 3 +$$

$$2(\lambda-1)\left(\left(\sum_{cyc} a\right)^2 + 3\right) + 2\lambda^2\left(\sum_{cyc} a - 3\right) - 8\lambda\left(\sum_{cyc} a\right)$$

$$\left(\because \sum_{cyc} ab \overset{AM-GM}{\geq} 3.\sqrt[3]{a^2b^2c^2} = 3\right) \overset{\substack{\lambda \geq 1 \\ and \\ \sum_{cyc} ab \geq 3}}{\geq}$$

$$6t + 9 - 3 + 2(\lambda-1)(t^2+3) + 2(t-3) - 8\lambda t \left(\begin{matrix} t = \sum_{cyc} a \text{ and } \because abc = 1 \text{ and} \\ \because t = \sum_{cyc} a \overset{AM-GM}{\geq} 3.\sqrt[3]{abc} = 3 \end{matrix}\right)$$

$$= 8t - 8\lambda t + 9 - 3 - 6 + 2(\lambda-1)(t^2+3) = 2(\lambda-1)(t^2+3-4t)$$

$$= 2(\lambda-1)(t-3)(t-1) \geq 0 (\because t \geq 3 \text{ and } \lambda \geq 1) \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{a+b}{a+b+\lambda} \geq \frac{6}{\lambda+2} \forall a, b, c > 0 \,\Big|\, abc = 1 \text{ and } \lambda \geq 1,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$