ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0, x+y+z=xyz$ and $\lambda\le 9$ then :

$$\lambda+x^2y^2z^2\ge\left(3+\frac{\lambda}{9}\right)\left(\sum_{cyc}xy\right)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$xyz=\sum_{cyc}x\overset{AM-GM}{\ge}3.\sqrt[3]{xyz}\Rightarrow x^2y^2z^2\ge 27\Rightarrow\left(\sum_{cyc}xy\right)^2\ge 3xyz\left(\sum_{cyc}x\right)$$
$$\overset{x+y+z=xyz}{=}3x^2y^2z^2\ge 81\Rightarrow\sum_{cyc}xy\ge 9\to(1)\ \text{and now,}$$

$$\lambda+x^2y^2z^2\overset{?}{\ge}\left(3+\frac{\lambda}{9}\right)\left(\sum_{cyc}xy\right)\Leftrightarrow x^2y^2z^2-3\sum_{cyc}xy\overset{?}{\ge}\frac{\lambda}{9}.\left(\sum_{cyc}xy-9\right)$$

$$\text{and now, since }\sum_{cyc}xy-9\overset{via\ (1)}{\ge}0\text{ and }\frac{\lambda}{9}\le 1\therefore\text{ it suffices to prove :}$$

$$x^2y^2z^2-3\sum_{cyc}xy\overset{?}{\ge}\sum_{cyc}xy-9\Leftrightarrow\left(\sum_{cyc}x\right)^2+\frac{9xyz}{\sum_{cyc}x}-4\sum_{cyc}xy\overset{?}{\ge}0$$

$$(\because x+y+z=xyz)\Leftrightarrow\left(\sum_{cyc}x\right)^3+9xyz-4\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\overset{?}{\ge}0$$

$$\Leftrightarrow\sum_{cyc}x^3+3xyz\overset{?}{\ge}\sum_{cyc}x^2y+\sum_{cyc}xy^2\to\text{true via Schur}$$

$$\therefore\lambda+x^2y^2z^2\ge\left(3+\frac{\lambda}{9}\right)\left(\sum_{cyc}xy\right)\ \forall x,y,z>0\,|\,x+y+z=xyz\ \text{and}\ \lambda\le 9,$$
$$''=''\ x=y=z=\sqrt{3}$$