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If $a, b, c > 0, \sum_{cyc} a^2b^2 = 1$, then :

$$\sum_{cyc}\sqrt{a^2+\frac{1}{a^2}} \geq \sum_{cyc} a + \frac{1}{abc}$$

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$$\sum_{cyc}\sqrt{a^2+\frac{1}{a^2}} \overset{?}{\geq} \sum_{cyc} a + \frac{1}{abc}$$

$$\Leftrightarrow \sum_{cyc} a^2 + \sum_{cyc}\frac{1}{a^2} + 2\sum_{cyc}\sqrt{\left(a^2+\frac{1}{a^2}\right)\left(b^2+\frac{1}{b^2}\right)} \underset{(*)}{\overset{?}{\geq}}$$

$$\left(\sum_{cyc} a\right)^2 + \frac{1}{a^2b^2c^2} + \frac{2}{abc}\left(\sum_{cyc} a\right)$$

$$\text{Now, LHS of } (*) \overset{\text{Reverse CBS}}{\geq} \sum_{cyc} a^2 + \sum_{cyc}\frac{1}{a^2} + 2\sum_{cyc} ab + 2\sum_{cyc}\frac{1}{ab}$$

$$= \left(\sum_{cyc} a\right)^2 + \frac{1}{a^2b^2c^2}\cdot\left(\sum_{cyc} a^2b^2\right) + \frac{2}{abc}\left(\sum_{cyc} a\right) \overset{\sum_{cyc} a^2b^2 = 1}{=}$$

$$\left(\sum_{cyc} a\right)^2 + \frac{1}{a^2b^2c^2} + \frac{2}{abc}\left(\sum_{cyc} a\right) \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\sqrt{a^2+\frac{1}{a^2}} \geq \sum_{cyc} a + \frac{1}{abc}\ \forall\, a, b, c > 0 \,\Big|\, \sum_{cyc} a^2b^2 = 1,$$

$$'' ='' \ \ a = b = c = \frac{1}{\sqrt[4]{3}}$$