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If $a, b, c > 0, a + b + c = 3$ and $n \in \mathbb{N}^*$ then :

$$\sum_{cyc} \frac{a^{n+1}}{b^{n+1}.c} \geq \frac{1}{3^n}\left(\sum_{cyc} ab^2\right)^{n+1}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$3 = \sum_{cyc} a \overset{AM-GM}{\geq} 3.\sqrt[3]{abc} \Rightarrow abc \leq 1 \rightarrow (1) \text{ and we have : } \sum_{cyc} ab^2 + \sum_{cyc} \frac{b}{a}$$

$$\overset{via\ (1)}{\leq} \frac{1}{abc}.\sum_{cyc} ab^2 + \frac{1}{abc}.\sum_{cyc} a^2b = \sum_{cyc} \frac{b}{c} + \sum_{cyc} \frac{a}{c} = \sum_{cyc} \frac{a}{b} + \sum_{cyc} \frac{b}{a}$$

$$\Rightarrow \sum_{cyc} \frac{a}{b} \geq \sum_{cyc} ab^2 \rightarrow (2) \text{ and now, since } n \in \mathbb{N}^* \therefore \sum_{cyc} \frac{a^{n+1}}{b^{n+1}.c} =$$

$$\sum_{cyc} \frac{\left(\frac{a}{b}\right)^{n+1}}{c} \overset{Holder}{\geq} \frac{\left(\sum_{cyc} \frac{a}{b}\right)^{n+1}}{3^{n-1}.\sum_{cyc} a} \overset{via\ (2)}{\geq} \frac{\left(\sum_{cyc} ab^2\right)^{n+1}}{3^{n-1}.\sum_{cyc} a} \overset{a+b+c=3}{=} \frac{1}{3^n}\left(\sum_{cyc} ab^2\right)^{n+1}$$

$$\therefore \sum_{cyc} \frac{a^{n+1}}{b^{n+1}.c} \geq \frac{1}{3^n}\left(\sum_{cyc} ab^2\right)^{n+1} \quad \forall\, a, b, c > 0 \,\middle|\, a + b + c = 3 \text{ and } n \in \mathbb{N}^*,$$

$'' ='' $ iff $a = b = c = 1$ (QED)