If $a,b,c>0, a+b+c=3$ and $n\in\mathbb{N}$ then :

$$\sum_{cyc}\left(\frac{a^3}{9-a(3b+3c+2bc)}\right)^n \geq 3$$

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$$\text{Firstly}, 9-a(3b+3c+2bc) \overset{a+b+c=3}{=} 9-2abc-3a(3-a)$$

$$=3a^2-9a+9-2abc \geq 3a^2-9a+9-2\left(\because 3=\sum_{cyc}a \overset{AM-GM}{\geq} 3.\sqrt[3]{abc}\right)>0$$

$$\because \Delta_{3a^2-9a+7}=81-84<0 \therefore 9-a(3b+3c+2bc) \text{ and analogs} >0 \to (1)$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius andinradius $=s,R,r$ (say);

then : $\sum_{cyc}a=s, abc=r^2s, \sum_{cyc}ab=4Rr+r^2, \sum_{cyc}a^2=s^2-8Rr-2r^2,$

$\sum_{cyc}a^3=s^3-12Rrs, \sum_{cyc}a^2b^2=r^2((4R+r)^2-2s^2)$ and then,

$$\sum_{cyc}\left(a^3(9-b(3c+3a+2ca))(9-c(3a+3b+2ab))\right)-$$

$$3\prod_{cyc}(9-a(3b+3c+2bc)) \boxed{=} \sum_{cyc}\left(a^3\left(9-2abc-(9b-3b^2)\right)\left(\begin{matrix}9-2abc-\\(9c-3c^2)\end{matrix}\right)\right)$$

$$-3\prod_{cyc}\left(9-2abc-(9a-3a^2)\right) \overset{a+b+c=3}{=} \frac{1}{9}\left(\left(\sum_{cyc}a\right)^3-6abc\right)^2\left(\sum_{cyc}a^3\right)-$$

$$\frac{1}{3}\left(\left(\sum_{cyc}a\right)^3-6abc\right)\left(\frac{9\left((\sum_{cyc}ab)(\sum_{cyc}a^2)-abc\sum_{cyc}a\right)(\sum_{cyc}a)^2}{9}-\frac{3\left((\sum_{cyc}a)(\sum_{cyc}a^2b^2)-abc\sum_{cyc}ab\right)(\sum_{cyc}a)}{3}\right)+$$

$$\frac{81abc(\sum_{cyc}a^2)(\sum_{cyc}a)^4}{81}-\frac{27abc\left((\sum_{cyc}a)(\sum_{cyc}ab)-3abc\right)(\sum_{cyc}a)^3}{27}+\frac{9a^2b^2c^2(\sum_{cyc}ab)(\sum_{cyc}a)}{3}-$$

$$3\left(\frac{\left((\sum_{cyc}a)^3-6abc\right)^3-\left((\sum_{cyc}a)^3-6abc\right)^2\left(3(\sum_{cyc}a)^2-3\sum_{cyc}a^2\right)(\sum_{cyc}a)}{27}\right)$$

$$-\left(\frac{\left(\sum_{cyc} a\right)^3 - 6abc}{3}\right)\left(\frac{81\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^4 - 81\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3abc\right)\left(\sum_{cyc} a\right)^3 + \left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a\right)^2}{27}\right)$$

$$+\frac{81abc(a+b)(b+c)(c+a)}{27}\left(\sum_{cyc} a\right)^3 \boxed{=} \frac{(s^3-6r^2s)^2(s^3-12Rrs)}{9} -$$

$$\frac{(s^3-6r^2s)\left(\begin{array}{c} s^2\left((4Rr+r^2)(s^2-8Rr-2r^2)-r^2s^2\right) - \\ s\left(sr^2((4R+r)^2-2s^2)-r^2s(4Rr+r^2)\right)\end{array}\right)}{3} + r^2s^5(s^2-8Rr-2r^2) -$$

$$r^2s^4(s(4Rr+r^2)-3r^2s)+3r^4s^3(4Rr+r^2)$$

$$-\frac{(s^3-6r^2s)^3-6s(4Rr+r^2)(s^3-6r^2s)^2}{9}$$

$$-(s^3-6r^2s)\left((4Rr+r^2)s^4-s^3(s(4Rr+r^2)-3r^2s)+s^2r^2((4R+r)^2-2s^2)\right)+$$

$$12Rr^3s^5 \overset{?}{\geq} 0 \Leftrightarrow 2s^4-(28Rr+19r^2)s^2+204Rr^3+159r^4 \boxed{\underset{(*)}{\overset{?}{\geq}}} 0 \text{ and } \because P =$$

$$(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove}:$$

$$\text{LHS of } (*) \overset{?}{\geq} P \Leftrightarrow (36R-39r)s^2 \boxed{\underset{(**)}{\overset{?}{\geq}}} r(512R^2-524Rr-109r^2)$$

$$\text{Again}, (36R-39r)s^2 \geq (36R-39r)\left(16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\right)$$

$$\left(\begin{array}{c}\because (R-r)(s^2-16Rr+5r^2) \overset{\text{Rouche}}{\geq} (R-r)\left(2R^2-6Rr+4r^2-2(R-2r)\sqrt{R^2-2Rr}\right) \\ = (R-2r)\left(\left(R-r-\sqrt{R^2-2Rr}\right)^2+r^2\right) \geq r^2(R-2r)\ \left(\because R-2r \overset{\text{Euler}}{\geq} 0\right)\end{array}\right)$$

$$\overset{?}{\geq} \text{RHS of } (**) \Leftrightarrow 64t^3-308t^2+473t-226 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow$$

$$(t-2)\left((t-2)(64t-52)+9\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \Rightarrow (**) \text{ is true} \overset{\text{via } ①}{\Rightarrow}$$

$$\sum_{cyc}\frac{a^3}{9-a(3b+3c+2bc)} \geq 3 \Rightarrow \sum_{cyc}\left(\frac{a^3}{9-a(3b+3c+2bc)}\right)^n \overset{\text{Holder}}{\geq}$$

$$3\left(\frac{\sum_{cyc}\frac{a^3}{9-a(3b+3c+2bc)}}{3}\right)^n \geq 3\ (\because n \in \mathbb{N}) \therefore \sum_{cyc}\left(\frac{a^3}{9-a(3b+3c+2bc)}\right)^n \geq 3$$

$$\forall\, a,b,c>0 \,\middle|\, a+b+c=3 \text{ and } n \in \mathbb{N}, \text{"="} \text{ iff } a=b=c=1 \text{ (QED)}$$