ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z, \lambda > 0, xyz = 1$ then :

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+(\lambda+1)(x+y+z)\geq 6\lambda+(2-\lambda)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

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Since $xyz = 1$, so we can assign $x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a}$ $(a, b, c > 0)$ and

$$\text{then : } \sum_{cyc}\frac{x}{y}+\sum_{cyc}x \overset{?}{\geq} 2\sum_{cyc}\frac{1}{x} \Leftrightarrow \sum_{cyc}\frac{ca}{b^2}+\sum_{cyc}\frac{a}{b} \underset{(*)}{\overset{?}{\geq}} 2\sum_{cyc}\frac{b}{a}$$

$$\text{Now, } \sum_{cyc}\frac{ca}{b^2}+\sum_{cyc}\frac{a}{b} = \sum_{cyc}\frac{ca}{b^2}+\sum_{cyc}\frac{c}{a} \overset{AM-GM}{\geq} 2.\sum_{cyc}\sqrt{\frac{ca}{b^2}.\frac{c}{a}} = 2\sum_{cyc}\frac{c}{b} = 2\sum_{cyc}\frac{b}{a}$$

$$\Rightarrow (*) \text{ is true} \therefore \sum_{cyc}\frac{x}{y}+\sum_{cyc}x \geq 2\sum_{cyc}\frac{1}{x} \to (1) \text{ and again, } \sum_{cyc}x+\sum_{cyc}\frac{1}{x} \overset{AM-GM}{\geq} 6$$

$$\Rightarrow \lambda\left(\sum_{cyc}x+\sum_{cyc}\frac{1}{x}-6\right) \geq 0 \ (\because \lambda > 0) \Rightarrow \lambda\sum_{cyc}x \geq 6\lambda - \lambda\sum_{cyc}\frac{1}{x} \to (2)$$

$$\therefore (1)+(2) \Rightarrow \frac{x}{y}+\frac{y}{z}+\frac{z}{x}+(\lambda+1)(x+y+z) \geq 6\lambda+(2-\lambda)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$\forall\, x, y, z, \lambda > 0 \,\big|\, xyz = 1, '' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$