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If $a, b, c > 0, a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ and $\lambda \ge 2$ then :

$$\sum_{cyc} \frac{1}{a^2 + \lambda} \le \frac{3}{\lambda + 1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{1}{a^2+\lambda} = \frac{\sum_{cyc} a^2b^2 + 2\lambda\sum_{cyc} a^2 + 3\lambda^2}{a^2b^2c^2 + \lambda\sum_{cyc} a^2b^2 + \lambda^2\sum_{cyc} a^2 + \lambda^3} \overset{?}{\le} \frac{3}{\lambda+1}$$

$$\Leftrightarrow (2\lambda - 1)\sum_{cyc} a^2b^2 + (\lambda^2 - 2\lambda)\sum_{cyc} a^2 + 3a^2b^2c^2 \overset{?}{\ge} 3\lambda^2$$

$$\Leftrightarrow \lambda^2\left(\sum_{cyc} a^2 - 3\right) + 2\lambda\left(\sum_{cyc} a^2b^2 - \sum_{cyc} a^2\right) \underset{(*)}{\overset{?}{\ge}} \sum_{cyc} a^2b^2 - 3a^2b^2c^2$$

$$\text{Now,} \sum_{cyc} ab = abc\sum_{cyc} a \le \frac{1}{3}\left(\sum_{cyc} ab\right)^2 \Rightarrow \sum_{cyc} ab \overset{①}{\ge} 3 \Rightarrow \left(\sum_{cyc} a\right)^2 \ge 9$$

$$\Rightarrow \sum_{cyc} a^2 \ge \frac{1}{3}\left(\sum_{cyc} a\right)^2 \ge 3 \text{ and so, } \lambda^2\left(\sum_{cyc} a^2 - 3\right) + 2\lambda\left(\sum_{cyc} a^2b^2 - \sum_{cyc} a^2\right)$$

$$\overset{\lambda \ge 2}{\ge} 2\lambda\left(\sum_{cyc} a^2 - 3 + \sum_{cyc} a^2b^2 - \sum_{cyc} a^2\right) \overset{\lambda \ge 2}{\ge} 4\left(\sum_{cyc} a^2b^2 - 3\right)$$

$$\left(\because \sum_{cyc} a^2b^2 \ge \frac{1}{3}\left(\sum_{cyc} ab\right)^2 \overset{\text{via } ①}{\ge} 3\right) \text{ and so, in order to prove } (*),$$

$$\text{it suffices to prove : } \sum_{cyc} a^2b^2 + a^2b^2c^2 \underset{(**)}{\overset{?}{\ge}} 4$$

Case 1 $abc \le 1$ and then : $\sum_{cyc} a^2b^2 + a^2b^2c^2 \overset{\text{Darij Grinberg}}{\ge}$

$$2abc\sum_{cyc} a - 1 - a^2b^2c^2 \overset{abc \le 1}{\ge} 2\sum_{cyc} ab - 2 \left(\because abc\sum_{cyc} a = \sum_{cyc} ab\right) \overset{\text{via } ①}{\ge} 6 - 2$$

$$= 4 \Rightarrow (**) \text{ is true}$$

Case 2 $abc > 1$ and then : $\sum_{cyc} a^2b^2 + a^2b^2c^2 \overset{abc>1}{>} \frac{1}{3}\left(\sum_{cyc} ab\right)^2 + 1 \overset{via\ ①}{\geq} 3 + 1$

$= 4 \Rightarrow (**)$ is true $\therefore$ combining both cases, $(**) \Rightarrow (*)$ is true $\therefore \sum_{cyc} \frac{1}{a^2+\lambda} \leq \frac{3}{\lambda+1}$

$\forall\, a, b, c > 0 \,\middle|\, a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ and $\lambda \geq 2$, " = " iff $a = b = c = 1$ (QED)