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If $a, b, c > 0, a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ and $\lambda \geq 2$ then :

$$\sum_{cyc} \frac{1}{1 + \lambda bc} \geq \frac{3}{\lambda + 1}$$

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$$\sum_{cyc} \frac{1}{1 + \lambda bc} = \sum_{cyc} \frac{a^2}{a^2 + \lambda a^2 bc} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a^2 + \lambda abc \sum_{cyc} a}$$

$$= \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a^2 + \lambda \sum_{cyc} ab} \left(\because \sum_{cyc} a = \sum_{cyc} \frac{1}{a}\right) \overset{?}{\geq} \frac{3}{\lambda + 1}$$

$$\Leftrightarrow \lambda \left(\sum_{cyc} a\right)^2 + \left(\sum_{cyc} a\right)^2 \overset{?}{\geq} 3 \sum_{cyc} a^2 + 3\lambda \sum_{cyc} ab$$

$$\Leftrightarrow \lambda \left(\sum_{cyc} a^2 - \sum_{cyc} ab\right) \overset{?}{\geq} 2 \left(\sum_{cyc} a^2 - \sum_{cyc} ab\right) \Leftrightarrow \frac{(\lambda - 2)}{2} . \left(\sum_{cyc} (a - b)^2\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because \lambda \geq 2 \therefore \sum_{cyc} \frac{1}{1 + \lambda bc} \geq \frac{3}{\lambda + 1} \ \forall a, b, c > 0 \mid a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

and $\lambda \geq 2$, $'' ='' $ iff $a = b = c = 1$ (QED)