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If $a, b, c > 0, abc = 1$ and $0 \le \lambda \le \frac{5}{3}$ then :

$$\sum_{cyc}\left(a+\frac{\lambda}{b}\right)^2 \ge \frac{3}{4}(\lambda+1)^2(a+b+c+1)$$

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$$\sum_{cyc}\left(a+\frac{\lambda}{b}\right)^2 = \sum_{cyc}a^2+\lambda^2\sum_{cyc}\frac{1}{a^2}+2\lambda\sum_{cyc}\frac{a}{b} \ge$$

$$\sum_{cyc}a^2+\frac{\lambda^2}{a^2b^2c^2}\sum_{cyc}a^2b^2+\frac{2\lambda}{\sqrt[3]{abc}}.\sum_{cyc}a \ge$$

$$\frac{1}{3}\left(\sum_{cyc}a\right)^2+\lambda^2 abc\sum_{cyc}a+2\lambda\left(\sum_{cyc}a\right)\ (\because abc=1)$$

$$=\frac{1}{3}\left(\sum_{cyc}a\right)^2+\lambda^2\sum_{cyc}a+2\lambda\left(\sum_{cyc}a\right)\ (\because abc=1) \overset{?}{\ge} \frac{3}{4}(\lambda+1)^2\left(\sum_{cyc}a+1\right)$$

$$\Leftrightarrow 4\left(\sum_{cyc}a\right)^2+12\lambda^2\sum_{cyc}a+24\lambda\left(\sum_{cyc}a\right) \overset{?}{\ge} (9\lambda^2+18\lambda+9)\left(\sum_{cyc}a+1\right)$$

$$\Leftrightarrow \left(\left(\sum_{cyc}a\right)^2-9\right)+3\left(\sum_{cyc}a\right)^2+12\lambda^2\sum_{cyc}a+24\lambda\left(\sum_{cyc}a\right) \overset{?}{\underset{(*)}{\ge}}$$

$$9\lambda^2+18\lambda+(9\lambda^2+18\lambda+9)\left(\sum_{cyc}a\right)$$

Now, since $\sum_{cyc}a \overset{AM-GM}{\ge} 3.\sqrt[3]{abc}=3 \Rightarrow \left(\sum_{cyc}a\right)^2-9 \ge 0$ **and hence,**

$3\left(\sum_{cyc}a\right)^2 \ge 9\sum_{cyc}a \therefore$ **in order to prove** $(*)$**, it suffices to prove :**

$$9\sum_{cyc}a+12\lambda^2\sum_{cyc}a+24\lambda\left(\sum_{cyc}a\right) \overset{?}{\ge} 9\lambda^2+18\lambda+(9\lambda^2+18\lambda+9)\left(\sum_{cyc}a\right)$$

$$\Leftrightarrow (3\lambda^2 + 6\lambda)\left(\sum_{cyc} a\right) \overset{?}{\geq} 9\lambda^2 + 18\lambda \to \text{true} \because \sum_{cyc} a \geq 3 \text{ (shown before) } and$$

$$\because \lambda \geq 0 \Rightarrow (*) \text{ is true} \therefore \sum_{cyc}\left(a + \frac{\lambda}{b}\right)^2 \geq \frac{3}{4}(\lambda + 1)^2(a + b + c + 1)$$

$$\forall\, a, b, c > 0 \,\middle|\, abc = 1 \text{ and } 0 \leq \lambda \leq \frac{5}{3}, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$